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SATELLITE MOTION IN AN AXI-SYMMETRIC FIELD, WITH AN APPLICATION TO LUNI-SOLAR PERTURBATIONS

by

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SUMMARY

The disturbing function for an axi-symmetric potential field is expanded into a doubly infinite Legendre harmonic series. The effect of the general harmonic term on the motion of a satellite in the field is developed by averaging with respect to the mean anomaly of the satellite and applying Lagrange's planetary equations. Expressions for the variation of orbital elements due to the main harmonics are listed, the results being applicable to the perturbations of an earth satellite due to asphericity of the earth and luni-solar influence. General expressions for secular and long-period perturbations are obtained.

Departmental Reference: Space 128

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1 INTRODUCTION

In attempting to describe the motion of an earth satellite the basic problem is to express analytically, and with sufficient accuracy, the deviation of the motion from the simple path that would be described if the only force acting were due to a uniform gravitating sphere. Three of the actual perturbing forces arise from the asphericity of the earth, from attraction by other bodies - notably the sun and moon - and from the pressure of solar radiation. The effects of these three sources of perturbation are studied in this Report. Other sources of perturbation, not studied here, include atmospheric drag and small forces due to electromagnetic drag and Einstein's modification of Newton's law of gravitation.

Since the first satellites were launched in 1957 a spate of papers on satellite perturbations has appeared. The effects of the earth's asphericity, of luni-solar attraction and of solar radiation have customarily been treated separately and there is a particularly large literature on the first of these. The object of the present paper is the development of a general analysis from which results applicable to all three sources of perturbations may be obtained.

Most writers of papers on satellite perturbations assume their readers have a wide background knowledge - in particular an understanding of the Legendre method of expanding the earth's potential. In the hope of making this Report more comprehensible to those who lack this background, and to make the account as complete as possible, the opportunity has been taken to give some of the fundamental material, mainly in the Appendices and sometimes rather unconventionally.

Merson[1] has investigated the motion of a satellite of a parent body whose gravitational field is axi-symmetric, by expanding the potential at a general exterior point as a series of terms involving Legendre polynomials and negative powers of the radial distance to the point from an origin on the axis of symmetry. Taking the earth to be a parent body of this type he has obtained the main effects of the first six parameters of asphericity - the J coefficients - on the orbital elements of an earth satellite.

Cook[2] has considered the perturbations of an earth satellite due to the attraction of a distant body (the sun or moon). He assumes the distant body does not move during one complete revolution of the satellite and, like Merson, describes the perturbations by giving the changes in the orbital elements of the satellite over such a revolution.

Now the disturbing field arising from a stationary body, considered as a point mass, is axi-symmetric. Hence the effect on satellite motion may be obtained by generalizing Merson's theory, using a series of terms, in the Legendre expansion of potential, with positive powers of the radial distance, Cook's results being for the first term in the series. Such a generalization is made in this Report, the potential being expressed as a doubly infinite series, powers of the radial distance running from $-\infty$ to $+\infty$. The effect of the general term of the series is studied. When results are applied to satellites of the earth, asphericity, lunar and solar perturbations must be dealt with separately since there is a different axis of symmetry associated with each (the earth's polar axis and axes pointing towards the moon and sun respectively). It is a straightforward matter, however, to transform results associated with one axis into the appropriate form relative to another axis and it is natural to regard the earth's polar axis as the fundamental reference.

Apart from a consideration of secular and long-period perturbations in section 10, the Report follows the Merson-Cook approach of studying perturbations over a complete revolution of a satellite. The treatment owes much to the work of Groves[3] and Kaula[4]. These authors give very general accounts of satellite motion in the earth's field, with axial symmetry no longer assumed and the short-period effect of every Legendre term included. In a subsequent paper[5] Kaula gives an equally general development of the lunar and solar disturbing functions. The very generality of these papers, however, involves two disadvantages: the analysis is formidable from the point of view of the average reader, and there is a dearth of concrete results at the end; algebraic expressions are given which cover all first-order terms (of the perturbations of the elements), but it is very difficult to write down specific terms from these expressions.

In this Report, therefore, the aim has been to make the general formulae as simple as possible and such that the generation of particular expressions is always an easy and obvious process. To the former end, and despite the extension to positive powers of the radial distance already mentioned, the formulae have been kept less general than those of Groves and Kaula in that short-period terms are averaged out as soon as possible, while the restriction to axially symmetric terms is itself an enormous simplification. The important short-period terms corresponding to J_2 only - i.e. to the dominant oblateness term in the case of the earth - are listed separately in section 10.1 of the paper. To assist in the derivation of particular results, each of the five basic functions defined in the Report is tabulated as far as the fourth

harmonic in the Legendre expansion. The functional expressions for higher harmonics are best obtained by the use of recurrence formulae; a list of the required formulae is given, generalizing some work of Merson[6].

From the formulae for the perturbations of an earth satellite after one complete revolution, expressions are derived for secular and long-period variations valid over many revolutions. Formulae for luni-solar perturbations, relevant to a not-too-distant satellite, are given for three situations: for a complete revolution of the satellite, the sun or moon being considered stationary; for the average over a revolution of the sun or moon; and for further averages with respect to revolutions of the perigee of the satellite and of its line of nodes relative to the line of nodes of the sun or moon.

2 PREVIOUS WORK

The considerable literature dealing with satellite perturbations has been remarked upon and some of the papers (Refs.1-6) mentioned. It may be of interest to give a brief survey of some of the more important contributions to this literature and we start with those which tackle the asphericity perturbations. Fairly detailed descriptions of, and comparisons between, the main theories are provided in useful papers by A. H. Cook[7] and Kaula[8].

First, historically, was a paper by King-Hele[9]. His results are limited mainly by the neglect of higher powers of the eccentricity and by the assumption that the existence could be postulated of a constantly inclined "orbital plane" in which the satellite would always lie. The theory of Brenner and Latta[10] provides a natural extension of King-Hele's methods, the characteristic of this method being its reliance on first principles rather than on the traditions of celestial mechanics.

Merson's theory[1] has provided the basis for all orbital analysis so far conducted at the R.A.E. He carries to its logical conclusion the idea of introducing non-osculating elements which, as far as J_2 is concerned, are as smooth as possible over one revolution of the satellite. The starting point is the set of six Lagrangian planetary equations for osculating elements, as it is also with the theories of several other authors including Zhongolovitch (see Ref.11) and Kozai[12]. Kozai's theory is the one used by the Smithsonian Astrophysical Observatory to analyse Baker-Nunn camera observations.

Kozai's paper is to be found in a noteworthy number of the Astronomical Journal which also gives the theories of Brouwer[13] and Garfinkel[14]. These authors have more advanced starting points which involve the methods of

Hamilton, Jacobi, Delaunay and Von-Zeipel. An intermediate orbit is required - the basic Kepler ellipse in the case of Brouwer, but a pseudo-ellipse, much closer to the true orbit, in the case of Garfinkel. Brouwer's theory is used by the National Aeronautics and Space Administration (NASA) as one basis for the derivation of orbits.

The classical method of Hansen has been modified by Musen[15,16] to give a theory which is part analytical, part numerical and is intended to facilitate the practical representation of satellite motion to very high accuracy. Musen's theory, also, has been used by NASA and formulae developed by Fisher[17] allow it to be compared directly with Brouwer's.

No survey would be complete without mention of the series of papers by Vinti of the U.S. Bureau of Standards. In the first[18] of the three main papers of this series Vinti introduced spheroidal co-ordinates in place of the usual spherical co-ordinates. By this means he obtained separability of the Hamilton-Jacobi equation for motion in a field which accounts for more than 99 per cent of the earth's asphericity. In terms of the J coefficients (see section 4.1) the field is such that $J_{2n} = (-1)^{n+1} J_2^n$ and $J_{2n+1} = 0$, and hence can represent the dominant J_2 coefficient completely, residual deviations being due to the values of $J_4 + J_2^2$, $J_6 - J_2^3$, etc. Vinti originally had a second free parameter in the potential so that J_3, say, could be represented by a choice of origin with $J_1 \neq 0$ and $J_{2n+1} = (-1)^n J_1 J_2^n$, but this idea was not pursued beyond the first paper. In the second main paper[19] the formal integrals from the first are evaluated, two possible sets of orbital elements defined and the exact solution for the intermediate orbit developed. In the third paper[20] Von Zeipel's method is used to obtain the perturbations, relative to the intermediate orbit, due to $J_4 + J_2^2$ and J_3.

On passing to the subject of luni-solar perturbations one finds far fewer papers. The main reasons for this are that for a close earth satellite the perturbations are much smaller than the asphericity perturbations and that the subject has less obvious novelty and so, perhaps, less intellectual attraction. Perturbations of the lunar orbit about the earth due to the sun and other disturbing bodies have, of course, been studied for many years and the theories of Hansen, Delaunay, Hill and Brown are described in such standard texts as Refs.21, 22 and 23. Although the problem of luni-solar perturbations of an earth satellite is superficially the same, it is actually different enough for most workers to have attacked it independently of the traditional theories.

The paper of Cook[2], already mentioned, considers only the main term in the Legendre expansion of the disturbing function and takes the disturbing body to be fixed. Allan[24], using vector methods, considers a further term in the expansion - the parallactic term - and Smith[25] yet another. Smith[25] also allows for the motion of the disturbing body and for long-period interaction with the secular effects of the earth's asphericity.

Kozai[26] develops the disturbing function for the effect of a disturbing body at a general position in its orbit, working with the main term only. Musen, Bailie and Upton[27] include the parallactic term in a similar development. Ref.27 includes short-period terms and requires 13 pages for the expression of the disturbing function in terms of orbital parameters. With secular and long-period terms only it still requires $6\frac{1}{2}$ pages; of these, $2\frac{1}{2}$ pages cover the main term in the Legendre expansion (Types 1 to 5 on pp. 25-27 of Ref.27) and agree with Kozai's development (pp. 8-9 of Ref.26) apart from a sign error in one of Kozai's terms (line 13 on page 9 of Ref.26).

(After the rest of the present paper was written, the author confirmed that the error in the disturbing function just referred to is indeed in Ref.26 and not in Ref.27. This was done by comparison with a third paper, recently written by Myrna Lewis[42]. Her expression for the disturbing function - equation (18) of Ref.42 - itself contains a copying error, which is not transmitted to the rest of the paper, in that the middle term of its third line should be multiplied by 2. Ref.42 goes beyond Refs.26 and 27 by giving a full development of secular and long-period perturbations to the orbital elements. Ref.27 gives no such development and Ref.26 gives only the secular terms for Ω and ω).

3 LAGRANGE'S PLANETARY EQUATIONS

We suppose that a satellite is moving in a gravitational field, the main effect of which is an inverse-square-law attraction towards a fixed point, O. Then if the field is conservative, the potential may be written $\mu/r + U$ where r is the distance from O and μ is a constant. The first term is from the central force and the second term, U, from the rest of the field. This term is called the disturbing function and we shall assume it to be a function of position but not of time. In the absence of perturbing force, the satellite would describe a fixed Kepler conic section orbit - we suppose an ellipse. The ellipse into which the actual orbit would "freeze" if, at time t, the perturbing force were instantaneously removed is called the osculating ellipse at time t. It can be specified by six elements; if the gravitational field is that of the earth (centre O) these are normally taken as the following set: semi-major

axis a, eccentricity e, inclination i, right ascension of the node Ω, argument of perigee ω and mean anomaly at the epoch (when $t = 0$) χ.

It is important to note, to avoid confusion, that χ - as with the other five elements - is defined for the current time t and varies - like the others - due to the perturbing force; the definition, however, involves an extension of the instantaneous "frozen" orbit back to $t = 0$. Use of this element has an inherent drawback. For if $\Delta\chi$ is the change in χ between two times t_1 and t_2, $\Delta\chi$ will depend on two extrapolations back to $t = 0$. But for an arbitrary displacement of the time origin (say to a system defined by $t' = t + \text{const}$), the extrapolations would be back to a different epoch ($t' = 0$) and so $\Delta\chi$ would have a different value. This is clearly most undesirable.

To avoid this difficulty we replace the element χ by another element, usually denoted by χ^* but here by σ for ease of notation, such that, if n is the mean motion given by $n^2 a^3 = \mu$,

$$\sigma + \int_0^t n \, dt = \chi + nt \quad , \tag{1}$$

where both sides of this equation give the mean anomaly, M, in the osculating orbit. The element σ itself suffers from the disadvantage that it is not, as is χ, immediately defined, for a given t, from the osculating orbit. By its use, however, it can be shown[23] that the equations for the variations of the elements are easier to integrate.

These equations, Lagrange's planetary equations[23], may now be stated in the following form:-

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial U}{\partial M} , \tag{2}$$

$$\frac{de}{dt} = \frac{1}{na^2 e}\left\{(1-e^2)\frac{\partial U}{\partial M} - (1-e^2)^{\frac{1}{2}}\frac{\partial U}{\partial \omega}\right\} , \tag{3}$$

$$\frac{di}{dt} = \frac{\operatorname{cosec} i}{na^2 (1-e^2)^{\frac{1}{2}}}\left\{\cos i \frac{\partial U}{\partial \omega} - \frac{\partial U}{\partial \Omega}\right\} , \tag{4}$$

$$\frac{d\Omega}{dt} = \frac{\operatorname{cosec} i}{na^2 (1-e^2)^{\frac{1}{2}}}\frac{\partial U}{\partial i} , \tag{5}$$

$$\frac{d\omega}{dt} = \frac{1}{na^2}\left\{\frac{(1-e^2)^{\frac{1}{2}}}{e}\frac{\partial U}{\partial e} - \frac{\cot i}{(1-e^2)^{\frac{1}{2}}}\frac{\partial U}{\partial i}\right\} , \tag{6}$$

$$\frac{d\sigma}{dt} = -\frac{1}{na^2}\left\{\frac{(1-e^2)}{e}\frac{\partial U}{\partial e} + 2a\frac{\partial U}{\partial a}\right\} . \tag{7}$$

In these equations the disturbing function U is assumed to be a function of a, e, i, Ω, ω and M, where

$$M = \sigma + \int_0^t n\, dt , \tag{8}$$

so that the dependence of n on a does not enter into $\frac{\partial U}{\partial a}$.

Once the equations have been set up there is no difficulty in picking out the main first-order terms of the solution. For we simply set a, e, i, Ω and ω constant on the right hand sides, after the partial derivatives have been formed. To eliminate short-period terms we then integrate over a complete revolution of M, i.e. over one orbit. (Short-period terms are included in the very general treatments of Groves[3] and Kaula[4]; for a satellite in the gravitational field of the earth, however, the only significant short-period terms arise from J_2 - defined in the next section - and these are given explicitly in section 10.1).

4 THE DISTURBING FUNCTION

4.1 Form assumed

The earth's gravitational potential may be expressed as

$$\frac{\mu}{r} + U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_n^m(\sin\beta)\,(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda)\right] . \quad \ldots . \quad (9)$$

This only differs from the form and notation recommended by the International Astronomical Union[28] in that U, in equation (9), is used just for the disturbing function and not for the entire potential. The derivation of the expression is outlined in Appendix A. The notation is as follows: $C_{n,m}$ and $S_{n,m}$ are numerical coefficients, r is the distance from the earth's centre of mass, R is the mean equatorial radius of the earth, β is latitude, λ is longitude measured east and $P_n^m(\sin\beta)$ is the associated Legendre function defined in Appendix B. The expansion is valid for points outside a sphere containing the total mass of the earth; neglecting the mass of the atmosphere, the radius of the smallest such sphere is only slightly in excess of R.

An important special case arises if the assumption is made that the gravitational field is symmetrical about the earth's polar axis. With the same deviation from the recommendations of the I.A.U.[28] as before, the potential may then be written

$$\frac{\mu}{r} + U = \frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\beta)\right] , \qquad (10)$$

where J_n is a numerical coefficient and $P_n(\sin\beta) = P_n^o(\sin\beta)$ is Legendre's polynomial. Equation (9) clearly reduces to equation (10) on taking $C_{n,o} = -J_n$ and all other C and S coefficients zero.

The assumption of polar symmetry gives a very good approximation to the actual field of the earth. It leads to considerable simplification in integrating Lagrange's equations. One reason for this is that the three 'orientating' elements in the set of orbital elements defined in section 3 - that is to say, i, Ω and ω - are defined relative to the assumed axis of symmetry (the polar axis) and the plane perpendicular to it (the equator).

In this paper, though we are considering perturbations associated with arbitrary central fields, the standard application is to the field of the earth and it is convenient to use the notation which refers to the earth. The restriction to axial symmetry is assumed so that the potential, outside a sphere containing the central mass, is described by equation (10). However, though the generalization to equation (9) is not made, we wish to generalize equation (10) another way. Gravitating mass is assumed to exist <u>outside</u> a

sphere of radius r_2 as well as inside a sphere of radius r_1. The centre of the two spheres is at O, a point on the axis of symmetry but not necessarily at the centre of mass of either region.

Then the potential may be taken to be given by

$$\frac{\mu}{r} + U = \frac{\mu}{r}\left[1 - \sum_{n=-\infty}^{\infty} J_n \left(\frac{R}{r}\right)^n P_\ell (\sin\beta)\right] , \tag{11}$$

where $\ell = n$ if $n > 0$, and $\ell = -n-1$ if $n < 0$; R is now an arbitrary fixed value of r. This expansion is valid at all points between the two spheres, i.e. for which $r_1 < r < r_2$. The terms with $n > 0$ arise from the interior mass and the terms with $n < 0$ from the exterior mass. A derivation of the expansion, extended to the case of axial non-symmetry, is given in Appendix A.

Three of the J coefficients may be eliminated very simply. Since P_o $(\sin\beta)$ is constant (+1) J_{-1} is associated with a constant contribution to U which may be disregarded. J_o will be zero so long as the constant μ is equal to GM_i, where G is the universal gravitational constant and M_i is the total interior mass; thus J_o may be eliminated by replacing μ, if necessary, by $\bar{\mu} = GM_i$. Finally, J_1 may be made zero by taking O at the centre of the interior mass. However, it is convenient to retain all terms in the expansion during the general development.

Thus the form assumed for the disturbing function is given by

$$U = \sum_{n=-\infty}^{\infty} U_n , \tag{12}$$

where

$$U_n = -\mu J_n R^n r^{-n-1} P_\ell (\sin\beta) . \tag{13}$$

4.2 <u>Development in terms of elements and mean anomaly</u>

Equation (13) gives U_n as a function of r and β (λ is missing because of the assumed axial symmetry) at all points of a certain region in space and hence at all points of any orbit that lies entirely within this region. In order to integrate Lagrange's equations, (2) to (7), it is necessary to replace this function of r and β by a function of the elements a, e, i, Ω, ω and of the mean anomaly M for the particular orbit. This we can do by expressing r and β as functions of a, e, i, Ω, ω and M by the usual relations for the Kepler ellipse. Two stages are required.

First we replace P_ℓ $(\sin\beta)$ by a function of i, ω and the true anomaly v. We follow Groves[3] with minor modifications to the notation and considerable simplification due to the assumed axial symmetry.

If γ is the angle measured from the north apex of the osculating orbit back along the orbit to the satellite we have

$$\gamma = \frac{\pi}{2} - (\omega + v) \tag{14}$$

by definition and

$$\sin\beta = \sin i \cos\gamma \tag{15}$$

by elementary spherical trigonometry. We now apply the addition theorem for Legendre polynomials which is given by equation (C1) of Appendix C, replacing n, m, ϕ, θ', θ and λ by ℓ, k, $\pi/2 - \beta$, $\pi/2$, i and γ respectively. This gives

$$P_\ell(\sin\beta) = \sum_{k=0}^{\ell} u_k \frac{(\ell-k)!}{(\ell+k)!} P_\ell^k(0)\, P_\ell^k(\cos i)\cos k\gamma \;, \tag{16}$$

where $u_k = 1$ if $k = 0$, and $u_k = 2$ if $k \neq 0$.

Now, from equation (B2) of Appendix B,

$$P_\ell^k(\cos i) = \sin^k i \, \frac{d^k P_\ell(\cos i)}{d(\cos i)^k} \;, \tag{17}$$

where the second (k'th derivative) factor is a polynomial in cos i. This polynomial does not vanish when cos i = 1 (unless $k > \ell$ when it disappears completely) and in fact

$$\frac{d^k P_\ell(\cos i)}{d(\cos i)^k} = \frac{(\ell+k)!}{2^k(\ell-k)!\,k!} \quad \text{when } i = 0 \;.$$

Hence it may be normalized - in a certain useful sense - and we define, for $k \leq \ell$,

$$A_\ell^k(i) = \frac{2^k(\ell-k)!\,k!}{(\ell+k)!} \, \frac{d^k P_\ell(\cos i)}{d(\cos i)^k} \;. \tag{18}$$

We now combine equations (13), (16), (17) and (18), writing

$$U_n = \sum_{k=0}^{\ell} U_{nk} \,, \tag{19}$$

where

$$U_{nk} = -\mu \, J_n \, R^n \, r^{-n-1} \, \frac{u_k \, P_\ell^k \,(0)}{2^k \; k!} \, \sin^k i \; A_\ell^k \,(i) \, \cos k\gamma \quad . \tag{20}$$

The function $A_\ell^k(i)$, which is 1 when $i = 0$, is conveniently expressed as a function of cos i and f, where

$$f = \sin^2 i \;.$$

Table 1 gives the particular functional expressions, for ℓ and k up to $\ell = 4$, $k = 4$. A recurrence relation is given in section 6.2 and derived in Appendix D.

In the second, and more complicated, stage of the replacement of r and β by a, e, i, Ω, ω and M we have to deal with the factors r^{-n-1} and $\cos k\gamma$ which occur in the expression of U_{nk} by equation (20). It is usual[3,4] at this point to appeal to Hansen's X function[22], by means of which one can write

$$r^{-n-1} \cos k\gamma = \sum_{s=-\infty}^{\infty} a^{-n-1} \, X_s^{-n-1,k} \,(e) \, \cos \left\{ sM + k \,(\omega - \tfrac{1}{2}\pi) \right\} \tag{21}$$

(remembering that $\gamma = \frac{1}{2}\pi - \omega - v$). Expression in terms of a, e, i, Ω, ω and M is clearly complete if this function is used.

In the present application, however, this approach is unnecessary, though the form of equation (21) helps to clarify the following argument. We have resolved to eliminate short-period terms in the solution of the Lagrangian equations, by integrating them (with respect to time) over a complete revolution of the satellite. But dt is proportional to dM, by equation (8). Then consideration of Lagrange's equations (2) to (7), in conjunction with the expression for U in terms of M using equation (21), shows that it is legitimate, and certainly simpler, just to integrate U with respect to M <u>before</u> substituting in the equations.

Now if U_{nk} (av) denotes the average value of U_{nk} over a complete revolution,

$$U_{nk}\ (\mathrm{av}) = \frac{1}{2\pi}\int_0^{2\pi} U_{nk}\ dM$$

$$= -\frac{\mu\ J_n\ R^n}{2\pi}\ \frac{u_k\ P_C^k(0)}{2^k\ k!}\ \sin^k i\ A_C^k(i)\int_0^{2\pi} r^{-n-1}\ \cos k\ (v+\omega-\tfrac{1}{2}\pi)\ dM$$

.... (22)

by equation (20). But for the Kepler ellipse,

$$dM = \left(\frac{r}{a}\right)^2\ (1-e^2)^{-\frac{1}{2}}\ dv$$

(from, for example, Chapter 2 of Ref.23). Hence, using the basic relation $p/r = 1+e\cos v$, where $p = a(1-e^2)$,

$$\int_0^{2\pi} r^{-n-1}\ \cos k\ (v+\omega-\tfrac{1}{2}\pi)\ dM = \frac{p^{-n+1}}{a^2(1-e^2)^{\frac{1}{2}}}\int_0^{2\pi} (1+e\cos v)^{n-1}\ \cos k\ (v+\omega-\tfrac{1}{2}\pi)\ dv\ .$$

.... (23)

Kaula[4] carries out this last integration for $n > 1$, but the integration is equally valid if $n < 0$. Using the binomial theorem for any index, positive or negative, and the relation

$$\cos^s v = 2^{-s}\sum_{q=0}^{s}\binom{s}{q}\cos\ (s-2q)\ v$$

for s a positive integer, we get

$$\int_0^{2\pi} (1+e\cos v)^{n-1}\ \cos k\ (v+\omega-\tfrac{1}{2}\pi)\ dv$$

$$= \int_0^{2\pi}\sum_{s=0}^{\infty}\binom{n-1}{s}\left(\frac{e}{2}\right)^s\sum_{q=0}^{s}\binom{s}{q}\cos\ (s-2q)\ v\ \cos k\ (v+\omega-\tfrac{1}{2}\pi)\ dv\ ,$$

.... (24)

where the series in s terminates if $n > 0$, and $\binom{n-1}{s}$ is defined for n positive, zero or negative by

$$\binom{n-1}{s} = (n-1)(n-2)\ ....\ (n-s)/s!\ .$$

Now, for any x,

$$\sum_{q=0}^{s} \binom{s}{q} \cos\,(s-2q)\,v \cos x = \tfrac{1}{2} \sum_{q=0}^{s} \binom{s}{q} \left[\cos\{x+(s-2q)\,v\} + \cos\{x-(s-2q)\,v\}\right]$$

$$= \sum_{q=0}^{s} \binom{s}{q} \cos\{x-(s-2q)\,v\} \quad , \tag{25}$$

since

$$\binom{s}{q} = \binom{s}{s-q} \; .$$

Also

$$\sum_{s=0}^{\infty} \sum_{q=0}^{s} = \sum_{q,s}^{0\leq q\leq s} = \sum_{q=0}^{\infty} \sum_{s=q}^{\infty} \; . \tag{26}$$

From equations (24), (25) and (26), with $x = k(v+\omega-\frac{1}{2}\pi)$,

$$\int_0^{2\pi} (1+e\cos v)^{n-1} \cos k\,(v+\omega-\tfrac{1}{2}\pi)\,dv$$

$$= \int_0^{2\pi} \sum_{q=0}^{\infty} \sum_{s=q}^{\infty} \binom{n-1}{s} \binom{s}{q} (\tfrac{1}{2}e)^s \cos\{k\,(\omega-\tfrac{1}{2}\pi) + (k-s+2q)\,v\}\,dv \; . \qquad \ldots\ldots (27)$$

But, of the double series on the right hand side, terms for which $k-s+2q \neq 0$ integrate to zero. Hence, since $s = k+2q$ satisfies $s > q$, equations (23) and (27) give

$$\int_0^{2\pi} r^{-n-1} \cos k\,(v+\omega-\tfrac{1}{2}\pi)\,dM = \frac{2\pi\,p^{-n+1}}{a^2(1-e^2)^{\frac{1}{2}}} \sum_{q=0}^{\infty} \binom{n-1}{k+2q} \binom{k+2q}{q} (\tfrac{1}{2}e)^{k+2q} \cos k\,(\omega-\tfrac{1}{2}\pi) \; . \qquad \ldots\ldots (28)$$

Now define

$$S_{nk}(e) = \sum_{q=0}^{\infty} \binom{n-1}{k+2q} \binom{k+2q}{q} (\tfrac{1}{2}e)^{2q} \quad . \tag{29}$$

Then from equations (22), (28) and (29),

$$U_{nk}(av) = -\mu J_n \left(\frac{R}{p}\right)^n \frac{u_k P_\ell^k(0)}{2^{2k} k!} \sin^k i \; A_\ell^k(i) \frac{e^k(1-e^2)^{\frac{1}{2}}}{a} S_{nk}(e) \cos k(\omega-\tfrac{1}{2}\pi) \quad . \quad .... \tag{30}$$

The series given by equation (29) terminates if $n \geq 1$, and $S_{nk}(e)$ is then a polynomial in e^2. But for this case $k \leq n$ (since $k \leq \ell$ and $\ell = n$). For $k = n$ the polynomial vanishes identically but otherwise it has a non-zero leading (constant) term, viz $\binom{n-1}{k}$. Hence it may be normalized and we define, for $n \geq 1$,

$$\left.\begin{aligned} B_n^k(e) &= S_{nk}(e)\Big/\binom{n-1}{k} \text{ if } 0 \leq k \leq n-1 \\ &\text{and} \\ B_n^n(e) &= 0 \text{ (though this is actually arbitrary).} \end{aligned}\right\} \tag{31}$$

A formula for $S_{nk}(e)$, when $n \geq 1$, in terms of $P_{n-1}^k\,(1/\sqrt{1-e^2})$ is given by equation (E13) of Appendix E.

For $n \leq 0$ the series is infinite. Fortunately it may be summed in terms of the $B_n^k(e)$ polynomial defined for $n \geq 1$. In Appendix E it is shown that

$$S_{nk}(e) = \binom{n-1}{k} (1-e^2)^{n-\frac{1}{2}} B_{-n+1}^k(e) \tag{32}$$

for $n \leq 0$ and $0 \leq k \leq \ell$ (where $\ell = -n-1$ if $n < 0$ and $\ell = n$ if $n = 0$).

Since it is desirable to retain a single formula for $U_{nk}(av)$ we seek to combine equations (31) and (32) before returning to equation (30). This can be done by taking the factors $(R/p)^n$ and $(1-e^2)^{\frac{1}{2}}$, from equation (30), with $S_{nk}(e)$ and writing

$$\left(\frac{R}{p}\right)^n (1-e^2)^{\frac{1}{2}} S_{nk}(e) = \binom{n-1}{k} \left(\frac{R}{b}\right)^n (1-e^2)^{\varepsilon} B_h^k(e) \quad , \tag{33}$$

where

$$b = p, \; h = n, \; \varepsilon = \tfrac{1}{2} \qquad \text{if } n \geq 1$$

and

$$b = a, \; h = -n+1, \; \varepsilon = 0 \qquad \text{if } n \leq 0 \quad .$$

The function B will now always be referred to with lower suffix h instead of n. Particular expressions for $B_h^k(e)$, up to h = 5, k = 5, are given in Table 2. A recurrence formula is given in section 6.2.

Before giving the final expression for $U_{nk}(av)$, it is convenient to combine together all the numerical factors, each a function of n and k only. We define

$$C_n^k = - \frac{u_k \, P_\ell^k(0)}{2^{2k} \; k!} \binom{n-1}{k} \tag{34}$$

for all integral n. The coefficient C_n^k is given in Table 3 for n running from -4 to +4 and, therefore, as far as k = 4. It is zero if ℓ-k is odd.

Equations (30), (33) and (34) now give

$$U_{nk}\,(av) = \mu \, J_n \, a^{-1} \left(\frac{R}{b}\right)^n B_h^k(e) \; e^k \; (1-e^2)^\varepsilon \; A_\ell^k\,(i) \; \sin^k i \; C_n^k \cos k\,(\omega - \tfrac{1}{2}\pi) \qquad \ldots\ldots \quad (35)$$

and this is the final result in the development of U. It is observed that $U_{nk}(av)$ is a function of a, e, i and ω only; Ω and M do not appear.

5 INTEGRATION OF THE PLANETARY EQUATIONS

Short-period terms having been eliminated from the disturbing function, correct first-order perturbations to the elements can only be obtained at intervals of one revolution. The integration of Lagrange's equations (2) to (7) over a complete revolution, from node to node or from perigee to perigee, is now quite trivial on taking $U_{nk}(av)$ from equations (12), (19) and (35). If Δζ, in the notation of Merson[1], indicates the change in a typical element ζ over one revolution,

$$\Delta\zeta = \frac{2\pi}{n} \frac{d\zeta}{dt}\,(av) \quad ,$$

where n, here, is the mean motion.

The (first-order) expression for each $\Delta\zeta$ can be written down as a double sum of terms with index n running from $-\infty$ to $+\infty$ and k from 0 to ℓ. For simplicity, however, we write

$$\Delta\zeta = 2\pi \sum_{n=-\infty}^{\infty} J_n \left(\frac{R}{b}\right)^n \zeta_n \tag{36}$$

and then, as in equation (19),

$$\zeta_n = \sum_{k=0}^{\ell} \zeta_{nk} \quad . \tag{37}$$

As a further simplification we introduce, as additional ζ_{nk}, the quantities ψ_{nk} and ρ_{nk} given by equations (44) and (45) which follow.

When substituting into equations (2) to (7), it must be remembered that b is a function of a and, when $n > 1$, also of e.

The expressions for the ζ_{nk} may now easily be written down as follows:-

$$a_{nk} = 0 \ , \tag{38}$$

$$e_{nk} = k\ C_n^k\ e^{k-1}\ (1-e^2)^{\varepsilon+\frac{1}{2}}\ \sin^k i\ A_\ell^k(i)\ B_h^k(e)\ \sin k\ (\omega-\tfrac{1}{2}\pi) \ , \tag{39}$$

$$i_{nk} = -\ e(1-e^2)^{-1}\ \cot i\ e_{nk} \ , \tag{40}$$

$$\Omega_{nk} = C_n^k\ e^k\ (1-e^2)^{\varepsilon-\frac{1}{2}}\ \sin^{k-2} i\ D_\ell^k(i)\ B_h^k(e)\ \cos k\ (\omega-\tfrac{1}{2}\pi) \ , \tag{41}$$

$$\omega_{nk} = -\ \Omega_{nk} \cos i + \psi_{nk} \ , \tag{42}$$

$$\sigma_{nk} = -\ (1-e^2)^{\frac{1}{2}}\ \psi_{nk} + \rho_{nk} \ , \tag{43}$$

$$\psi_{nk} = C_n^k\ e^{k-2}\ (1-e^2)^{\frac{1}{2}-\varepsilon}\ \sin^k i\ A_\ell^k(i)\ E_n^k(e)\ \cos k\ (\omega-\tfrac{1}{2}\pi) \ , \tag{44}$$

$$\rho_{nk} = 2(n+1)\ C_n^k\ e^k\ (1-e^2)^{\varepsilon}\ \sin^k i\ A_\ell^k(i)\ B_h^k(e)\ \cos k\ (\omega-\tfrac{1}{2}\pi) \ , \tag{45}$$

where $D_\ell^k(i)$ and $E_n^k(e)$ are defined by

$$D_\ell^k(i) \quad = \quad k \cos i \; A_\ell^k(i) \quad + \quad \sin i \frac{d}{di} A_\ell^k(i) \quad , \tag{46}$$

$$E_n^k(e) \quad = \quad \{k + (2n-k-1)e^2\} \; B_h^k(e) \quad + \quad e(1-e^2) \frac{d}{de} B_h^k(e) \text{ if } n > 1 \tag{47}$$

and

$$E_n^k(e) \quad = \quad k \; B_h^k(e) \quad + \quad e \frac{d}{de} B_h^k(e) \quad \text{if } n < 0 \quad . \tag{48}$$

We observe that $D_\ell^k(0) = E_n^k(0) = k$. Thus in particular $D_\ell^0(0) = E_n^0(0) = 0$ and this is why these functions can not be normalised in the way that $A_\ell^k(i)$ and $B_h^k(e)$ were. The function $D_\ell^k(i)$ is given in Table 4 as far as $\ell = 4$, $k = 4$, and $E_n^k(e)$ in Table 5 for n running from -4 to +4 and as far as k = 4.

6 EXPLICIT EXPRESSIONS FOR THE ζ_{nk}

6.1 The special cases n = -1, 0 and +1

These special cases have been mentioned briefly in section 4.1 when the final expression for the disturbing potential was introduced. It may now be seen from equations (38) to (45) that the ζ_n have their right values for these cases. Thus all ζ_n are zero except that

$$\sigma_o \quad = \quad \rho_o \quad = -2 \quad .$$

For $n = -1$, $U_{-1} = -\dfrac{\mu J_{-1}}{R}$. This is constant and so may be disregarded.

For $n = 0$, $U_o = -\dfrac{\mu J_o}{r}$; i.e. with no other disturbing term present, the full potential is

$$\frac{\mu}{r} + U_o \quad = \quad \frac{\mu(1-J_o)}{r} \quad .$$

Thus the perturbation may be combined with the central force itself, using - instead of μ - an effective $\bar{\mu} = \mu(1-J_o)$. An orbit in the field of force may be described either by constant 'barred' elements based on $\bar{\mu}$ or else by the normal osculating elements based on μ. In the latter case we must tolerate variations in the elements, due to the perturbation.

Although the normal osculating elements are not all constant it is clear - from the fact that the orbit really is an ellipse - that the changes in a, e, i, Ω and ω over a complete revolution must be zero. For σ, however, the situation is different. We note that (varying) a is related to (constant) $\bar{a}$ by the equation

$$\mu\left(\frac{2}{r} - \frac{1}{a}\right) = \bar{\mu}\left(\frac{2}{r} - \frac{1}{\bar{a}}\right) ,$$

since both sides give the square of the velocity. Thus the osculating mean motion, n, varies during a revolution. Since the last equation leads to

$$\frac{1}{a} - \frac{1}{\bar{a}} = J_o\left(\frac{2}{r} - \frac{1}{\bar{a}}\right) , \qquad (49)$$

it follows fairly easily that the mean value of n, the 'mean mean motion', exceeds $\bar{n}$ (where $\bar{n}^2 \bar{a}^3 = \bar{\mu}$) by $2 J_o n$, to the first order. Then equation (8) leads at once to the secular change given by $\sigma_o = -2$.

The final special case is for index n = +1. Then $U_1 = -\dfrac{\mu J_1 R \sin\beta}{r^2}$ and, with no other disturbing term present, the full potential, $\mu/r + U_1$, is to first order the same as

$$\frac{\mu}{r}\sum_{q=0}^{\infty}\left(\frac{-J_1 R}{r}\right)^q P_q(\sin\beta) = \frac{\mu}{(r^2 + J_1^2 R^2 + 2 r J_1 R \sin\beta)^{\frac{1}{2}}} .$$

Hence, using equation (A1) of Appendix A, the potential is essentially the same as for an inverse-square-law force directed towards the point $r = -J_1 R$, $\beta = \frac{1}{2}\pi$, instead of towards the origin. It follows that there is no change, to first order, in any of the osculating elements over a complete revolution. If the special point $r = -J_1 R$, $\beta = \frac{1}{2}\pi$ is used as a new origin, the corresponding new value of J_1 becomes zero. That the new origin is in fact the centroid of interior mass is easily seen by considering equation (A7) of Appendix A with the first moment, $\int_{(\text{matter})} r' \cos\theta' = C_1^o$, equal to zero.

6.2 Recurrence relations

The functions A, B, C, D and E, used in equations (38) to (45), become so complicated for numerically large values of the index n that it is very laborious - and indeed dangerous - to work out their expansions from first principles each time. Now the functions are based on Legendre functions, for which there exist well-known recurrence relations. Hence it is logical to look for such relations for the functions A, B, C, D and E.

Recurrence relations are developed in Appendix D. The complete set of results is given here. It should be noted that all but two of the relations are for the upper suffix (k) fixed.

$\underline{A^k_\ell(i)}$ For given $k \geq 0$ and for each $\ell \geq k + 2$,

$$A^k_\ell(i) = \frac{2\ell-1}{\ell+k} \cos i \, A^k_{\ell-1}(i) - \frac{\ell-k-1}{\ell+k} A^k_{\ell-2}(i) \quad . \tag{50}$$

The starting values are given by

$$A^k_k(i) = 1 \qquad \text{and} \qquad A^k_{k+1}(i) = \cos i \quad .$$

$\underline{B^k_h(e)}$ For given $k \geq 0$ and for each $h \geq k + 3$,

$$B^k_h(e) = \frac{2h-3}{h+k-1} B^k_{h-1}(e) - \frac{h-k-2}{h+k-1} (1-e^2) B^k_{h-2}(e) \quad . \tag{51}$$

The starting values are given by

$$B^k_{k+1}(e) = 1 \qquad \text{and} \qquad B^k_{k+2}(e) = 1 \quad ;$$

in addition $B^k_k(e)$, for $k \geq 1$, may be (arbitrarily) taken as zero.

$\underline{C^k_n}$ (i) $n \geq 0$. For given $k \geq 0$ and for each $n \geq k + 3$,

$$C^k_n = - \frac{(n-1)(n-2)(n+k-1)}{(n-k)(n-k-1)(n-k-2)} C^k_{n-2} \quad ; \tag{52}$$

also, for $k \geq 1$,

$$C^k_{k+2} = \frac{(k+1)(2k+1)}{4k^2} \frac{u_k}{u_{k-1}} C^{k-1}_{k+1} \quad , \tag{53}$$

with u_k as defined immediately after equation (16).

The starting values are given by

$C^0_2 = \frac{1}{2}$ and $C^k_{k+1} = 0$ for $k \geq 0$; in addition, $C^0_0 = -1$ and $C^k_k = 0$ if $k \neq 0$.

(ii) $n < 0$. For given $k \geq 0$ and for each n with $-n \geq k + 3$,

$$C^k_n = - \frac{(-n+k-2)(-n+k-1)(-n+k)}{(-n)(-n-1)(-n-k-1)} C^k_{n+2} \quad ; \tag{54}$$

also, for $k > 1$,

$$C^{k}_{-k-1} = -\frac{4k^2-1}{2k(k+1)} \frac{u_k}{u_{k-1}} C^{k-1}_{-k} \quad . \tag{55}$$

The starting values are given by

$$C^{k}_{-k-2} = 0 \qquad \text{and} \qquad C^{o}_{-1} = -1 \quad .$$

$D^k_\ell(i)$.

There is no three-term recurrence relation when $k > 1$.

However the case $k = 0$ is the most important one and for this case, for each $\ell > 2$,

$$D^{o}_{\ell}(i) = \frac{2\ell-1}{\ell-1} \cos i \, D^{o}_{\ell-1}(i) - \frac{\ell}{\ell-1} D^{o}_{\ell-2}(i) \quad . \tag{56}$$

The starting values are given by

$$D^{o}_{o}(i) = 0 \qquad \text{and} \qquad D^{o}_{1}(i) = -f \quad ,$$

where $f = \sin^2 i$. It follows immediately that the leading term of $D^{o}_{\ell}(i)$ is $-\frac{1}{2}\, \ell(\ell+1)f \cos i$ if ℓ is even and $-\frac{1}{2}\, \ell(\ell+1)f$ if ℓ is odd.

$E^k_n(e)$

Again, there is no three-term recurrence relation unless $k = 0$; for this case, and for each $n > 2$,

$$E^{o}_{n}(e) = \frac{2n-1}{n-1} E^{o}_{n-1}(e) - \frac{n}{n-1} (1-e^2) E^{o}_{n-2}(e) \quad . \tag{57}$$

The starting values are given by

$$E^{o}_{o}(e) = 0 \qquad \text{and} \qquad E^{o}_{1}(e) = e^2 \quad .$$

It follows that the leading term of $E^{o}_{n}(e)$ is $\frac{1}{2}\, n(n+1)e^2$.

For $n < 0$, $E^{o}_{n}(e)$ is given at once - see equation (E23) of Appendix E - by

$$E^{o}_{n}(e) = E^{o}_{-n-1}(e) \quad . \tag{58}$$

6.3 ζ_{nk} for $n > 2$

As follows from section 4.1 the terms U_n, with $n > 2$, of the disturbing function relate at once to the problem of satellite motion about an axi-symmetric body. In the case of the earth, the coefficient J_2 is of order 10^{-3} and J_3, J_4 etc are of order 10^{-6}.

Merson[1] has given e_n, i_n, Ω_n, ω_n and σ_n for n = 2 to 6 inclusive; i.e. he has given synthetic expressions for the ζ_n from which the ζ_{nk} can at once be picked out. However, Ref.1 gives no general procedure for writing down ζ_{nk}.

Such a general procedure is now available from the recurrence relations of section 6.2. These relations and the basic equations (38) to (45) have been used to obtain the complete set of non-zero ζ_{nk}, extended up to n = 9, which follows. For convenience, i_{nk}, ω_{nk} and σ_{nk} are not given, since they may at once be obtained from e_{nk}, ψ_{nk} and ρ_{nk}, using equations (40), (42) and (43). It is observed that non-zero ζ_{nk} can only occur when n-k is even and $0 < k < n-2$. In every case, of course, a_{nk} is zero; e_{nk} - and hence i_{nk} - is zero for k = 0 (no secular terms occur for the eccentricity or inclination).

<u>n = 2</u> $\quad \Omega_{20} = -\frac{3}{2}\cos i, \quad \psi_{20} = \frac{3}{2}\left(1-\frac{3}{2}f\right), \quad \rho_{20} = 3\,(1-e^2)^{\frac{1}{2}}\left(1-\frac{3}{2}f\right)$.

<u>n = 3</u>

$$e_{31} = -\frac{3}{2}(1-e^2)\left(1-\frac{5}{4}f\right)\sin i\cos\omega,$$

$$\Omega_{31} = \frac{3}{2}e\left(1-\frac{15}{4}f\right)\cot i\sin\omega\ ,$$

$$\psi_{31} = \frac{3}{2}e^{-1}\,(1+4\,e^2)\left(1-\frac{5}{4}f\right)\sin i\sin\omega\ ,$$

$$\rho_{31} = 12e\,(1-e^2)^{\frac{1}{2}}\left(1-\frac{5}{4}f\right)\sin i\sin\omega\ .$$

<u>n = 4</u>

$$\Omega_{40} = \frac{15}{4}\left(1+\frac{3}{2}e^2\right)\left(1-\frac{7}{4}f\right)\cos i,$$

$$\psi_{40} = -\frac{15}{4}\left(1+\frac{3}{4}e^2\right)\left(1-5\,f+\frac{35}{8}f^2\right)\ ,$$

$$\rho_{40} = -\frac{15}{4}(1-e^2)^{\frac{1}{2}}\left(1+\frac{3}{2}e^2\right)\left(1-5\,f+\frac{35}{8}f^2\right)\ .$$

$$e_{42} = -\frac{45}{16}\, e\, (1-e^2)\, f \left(1 - \frac{7}{6} f\right) \sin 2\omega \ ,$$

$$\Omega_{42} = -\frac{45}{16}\, e^2 \left(1 - \frac{7}{3} f\right) \cos i \cos 2\omega \ ,$$

$$\psi_{42} = -\frac{45}{16} \left(1 + \frac{5}{2} e^2\right) f \left(1 - \frac{7}{6} f\right) \cos 2\omega \ ,$$

$$\rho_{42} = -\frac{225}{16}\, e^2\, (1-e^2)^{\frac{1}{2}}\, f \left(1 - \frac{7}{6} f\right) \cos 2\omega \ .$$

<u>n = 5</u>

$$e_{51} = \frac{15}{4}\, (1-e^2) \left(1 + \frac{3}{4} e^2\right)\left(1 - \frac{7}{2} f + \frac{21}{8} f^2\right) \sin i \cos \omega \ ,$$

$$\Omega_{51} = -\frac{15}{4}\, e \left(1 + \frac{3}{4} e^2\right)\left(1 - \frac{21}{2} f + \frac{105}{8} f^2\right) \cot i \sin\omega \ ,$$

$$\psi_{51} = -\frac{15}{4}\, e^{-1} \left(1 + \frac{41}{4} e^2 + \frac{9}{2} e^4\right)\left(1 - \frac{7}{2} f + \frac{21}{8} f^2\right) \sin i \sin \omega \ ,$$

$$\rho_{51} = -45\, e(1-e^2)^{\frac{1}{2}} \left(1 + \frac{3}{4} e^2\right)\left(1 - \frac{7}{2} f + \frac{21}{8} f^2\right) \sin i \sin \omega \ .$$

$$e_{53} = \frac{105}{32}\, e^2\, (1-e^2)\, f \left(1 - \frac{9}{8} f\right) \sin i \cos 3\omega \ ,$$

$$\Omega_{53} = -\frac{105}{32}\, e^3 \left(1 - \frac{15}{8} f\right) \sin i \cos i \sin 3\omega \ ,$$

$$\psi_{53} = -\frac{105}{32}\, e(1+2e^2)\, f \left(1 - \frac{9}{8} f\right) \sin i \sin 3\omega \ ,$$

$$\rho_{53} = -\frac{105}{8}\, e^3\, (1-e^2)^{\frac{1}{2}}\, f \left(1 - \frac{9}{8} f\right) \sin i \sin 3\omega \ .$$

<u>n = 6</u>

$$\Omega_{60} = -\frac{105}{16} \left(1 + 5e^2 + \frac{15}{8} e^4\right)\left(1 - \frac{9}{2} f + \frac{33}{8} f^2\right) \cos i \ ,$$

$$\psi_{60} = \frac{105}{16} \left(1 + \frac{5}{2} e^2 + \frac{5}{8} e^4\right)\left(1 - \frac{21}{2} f + \frac{189}{8} f^2 - \frac{231}{16} f^3\right) \ ,$$

$$\rho_{60} = \frac{35}{8}\, (1-e^2)^{\frac{1}{2}} \left(1 + 5e^2 + \frac{15}{8} e^4\right)\left(1 - \frac{21}{2} f + \frac{189}{8} f^2 - \frac{231}{16} f^3\right) \ .$$

$$e_{62} = \frac{525}{32} e(1-e^2)\left(1+\frac{1}{2}e^2\right) f \left(1-3f+\frac{33}{16}f^2\right) \sin 2\omega ,$$

$$\Omega_{62} = \frac{525}{32} e^2\left(1+\frac{1}{2}e^2\right)\left(1-6f+\frac{99}{16}f^2\right) \cos i \cos 2\omega ,$$

$$\psi_{62} = \frac{525}{32}\left(1+\frac{11}{2}e^2+\frac{7}{4}e^4\right) f \left(1-3f+\frac{33}{16}f^2\right) \cos 2\omega ,$$

$$\rho_{62} = \frac{3675}{32} e^2 (1-e^2)^{\frac{1}{2}}\left(1+\frac{1}{2}e^2\right) f \left(1-3f+\frac{33}{16}f^2\right) \cos 2\omega .$$

$$e_{64} = \frac{1575}{512} e^3 (1-e^2) f^2 \left(1-\frac{11}{10}f\right) \sin 4\omega ,$$

$$\Omega_{64} = \frac{1575}{512} e^4 f \left(1-\frac{33}{20}f\right) \cos i \cos 4\omega ,$$

$$\psi_{64} = \frac{1575}{512} e^2 \left(1+\frac{7}{4}e^2\right) f^2 \left(1-\frac{11}{10}f\right) \cos 4\omega ,$$

$$\rho_{64} = \frac{11025}{1024} e^4 (1-e^2)^{\frac{1}{2}} f^2 \left(1-\frac{11}{10}f\right) \cos 4\omega .$$

$\underline{n = 7}$

$$e_{71} = -\frac{105}{16}(1-e^2)\left(1+\frac{5}{2}e^2+\frac{5}{8}e^4\right)\left(1-\frac{27}{4}f+\frac{99}{8}f^2-\frac{429}{64}f^3\right) \sin i \cos \omega ,$$

$$\Omega_{71} = \frac{105}{16} e \left(1+\frac{5}{2}e^2+\frac{5}{8}e^4\right)\left(1-\frac{81}{4}f+\frac{495}{8}f^2-\frac{3003}{64}f^3\right) \cot i \sin \omega ,$$

$$\psi_{71} = \frac{105}{16} e^{-1}\left(1+\frac{39}{2}e^2+\frac{225}{8}e^4+5e^6\right)\left(1-\frac{27}{4}f+\frac{99}{8}f^2-\frac{429}{64}f^3\right) \sin i \sin \omega ;$$

$$\rho_{71} = 105\, e(1-e^2)^{\frac{1}{2}}\left(1+\frac{5}{2}e^2+\frac{5}{8}e^4\right)\left(1-\frac{27}{4}f+\frac{99}{8}f^2-\frac{429}{64}f^3\right) \sin i \sin \omega .$$

$$e_{73} = -\frac{4725}{128} e^2(1-e^2)\left(1+\frac{3}{8}e^2\right) f \left(1-\frac{11}{4}f+\frac{143}{80}f^2\right) \sin i \cos 3\omega ,$$

$$\Omega_{73} = \frac{4725}{128} e^3 \left(1+\frac{3}{8}e^2\right)\left(1-\frac{55}{12}f+\frac{1001}{240}f^2\right) \sin i \cos i \sin 3\omega ,$$

$$\psi_{73} = \frac{4725}{128} e \left(1+\frac{95}{24}e^2 \cdot e^4\right) f \left(1-\frac{11}{4}f+\frac{143}{80}f^2\right) \sin i \sin 3\omega ,$$

$$\rho_{73} = \frac{1575}{8} e^3(1-e^2)^{\frac{1}{2}}\left(1+\frac{3}{8}e^2\right) f \left(1-\frac{11}{4}f+\frac{143}{80}f^2\right) \sin i \sin 3\omega .$$

$$e_{75} = -\frac{10395}{4096}\, e^4(1-e^2)\, f^2 \left(1-\frac{13}{12}\, f\right) \sin i \cos 5\omega \ ,$$

$$\Omega_{75} = \frac{10395}{4096}\, e^5\, f \left(1-\frac{91}{60}\, f\right) \sin i \cos i \sin 5\omega \ ,$$

$$\psi_{75} = \frac{10395}{4096}\, e^3 \left(1+\frac{8}{5}\, e^2\right) f^2 \left(1-\frac{13}{12}\, f\right) \sin i \sin 5\omega \ ,$$

$$\rho_{75} = \frac{2079}{256}\, e^5(1-e^2)^{\frac{1}{2}}\, f^2 \left(1-\frac{13}{12}\, f\right) \sin i \sin 5\omega \ .$$

<u>n = 8</u>

$$\Omega_{80} = \frac{315}{32}\left(1+\frac{21}{2}\, e^2+\frac{105}{8}\, e^4+\frac{35}{16}\, e^6\right)\left(1-\frac{33}{4}\, f+\frac{143}{8}\, f^2-\frac{715}{64}\, f^3\right) \cos i \ ,$$

$$\psi_{80} = -\frac{315}{32}\left(1+\frac{21}{4}\, e^2+\frac{35}{8}\, e^4+\frac{35}{64}\, e^6\right)\left(1-18\, f+\frac{297}{4}\, f^2-\frac{429}{4}\, f^3+\frac{6435}{128}\, f^4\right),$$

$$\rho_{80} = -\frac{315}{64}\,(1-e^2)^{\frac{1}{2}}\left(1+\frac{21}{2}\, e^2+\frac{105}{8}\, e^4+\frac{35}{16}\, e^6\right)\left(1-18\, f+\frac{297}{4}\, f^2-\frac{429}{4}\, f^3 + \frac{6435}{128}\, f^4\right).$$

$$e_{82} = -\frac{6615}{128}\, e(1-e^2)\left(1+\frac{5}{3}\, e^2+\frac{5}{16}\, e^4\right) f \left(1-\frac{11}{2}\, f+\frac{143}{16}\, f^2-\frac{143}{32}\, f^3\right) \sin 2\omega \ ,$$

$$\Omega_{82} = -\frac{6615}{128}\, e^2\left(1+\frac{5}{3}\, e^2+\frac{5}{16}\, e^4\right)\left(1-11\, f+\frac{429}{16}\, f^2-\frac{143}{8}\, f^3\right) \cos i \cos 2\omega \ ,$$

$$\psi_{82} = -\frac{6615}{128}\left(1+\frac{59}{6}\, e^2+\frac{485}{48}\, e^4+\frac{45}{32}\, e^6\right) f \left(1-\frac{11}{2}\, f+\frac{143}{16}\, f^2-\frac{143}{32}\, f^3\right) \cos 2\omega \ ,$$

$$\rho_{82} = -\frac{59535}{128}\, e^2(1-e^2)^{\frac{1}{2}}\left(1+\frac{5}{3}\, e^2+\frac{5}{16}\, e^4\right) f \left(1-\frac{11}{2}\, f+\frac{143}{16}\, f^2-\frac{143}{32}\, f^3\right) \cos 2\,\omega \ .$$

$$e_{84} = -\frac{121275}{2048}\, e^3(1-e^2)\left(1+\frac{3}{10}e^2\right) f^2 \left(1-\frac{13}{5}\, f+\frac{13}{8}\, f^2\right) \sin 4\,\omega \ ,$$

$$\Omega_{84} = -\frac{121275}{2048}\, e^4\left(1+\frac{3}{10}\, e^2\right) f \left(1-\frac{39}{10}\, f+\frac{13}{4}\, f^2\right) \cos i \cos 4\,\omega \ ,$$

$$\psi_{84} = -\frac{121275}{2048}\, e^2\left(1+\frac{16}{5}\, e^2+\frac{27}{40}\, e^4\right) f^2 \left(1-\frac{13}{5}\, f+\frac{13}{8}\, f^2\right) \cos 4\,\omega \ ,$$

$$\rho_{84} = -\frac{1091475}{4096}\, e^4(1-e^2)^{\frac{1}{2}}\left(1+\frac{3}{10}\, e^2\right) f^2 \left(1-\frac{13}{5}\, f+\frac{13}{8}\, f^2\right) \cos 4\,\omega \ .$$

$$e_{86} = -\frac{63063}{32768}\, e^5(1-e^2)\, f^3 \left(1 - \frac{15}{14} f\right) \sin 6\,\omega \quad ,$$

$$\Omega_{86} = -\frac{63063}{32768}\, e^6\, f^2 \left(1 - \frac{10}{7} f\right) \cos i \cos 6\,\omega \quad ,$$

$$\psi_{86} = -\frac{63063}{32768}\, e^4 \left(1 + \frac{3}{2} e^2\right) f^3 \left(1 - \frac{15}{14} f\right) \cos 6\,\omega \quad ,$$

$$\rho_{86} = -\frac{189189}{32768}\, e^6(1-e^2)^{\frac{1}{2}}\, f^3 \left(1 - \frac{15}{14} f\right) \cos \;\; \omega \quad .$$

<u>n = 9</u>

$$e_{91} = \frac{315}{32}(1-e^2)\left(1 + \frac{21}{4} e^2 + \frac{35}{8} e^4 + \frac{35}{64} e^6\right)$$
$$\times \left(1 - 11\, f + \frac{143}{4} f^2 - \frac{715}{16} f^3 + \frac{2431}{128} f^4\right) \sin i \cos \omega \quad ,$$

$$\Omega_{91} = -\frac{315}{32}\, e \left(1 + \frac{21}{4} e^2 + \frac{35}{8} e^4 + \frac{35}{64} e^6\right)$$
$$\times \left(1 - 33\, f + \frac{715}{4} f^2 - \frac{5005}{16} f^3 + \frac{21879}{128} f^4\right) \cot i \sin \omega \quad ,$$

$$\psi_{91} = -\frac{315}{32}\, e^{-1} \left(1 + \frac{127}{4} e^2 + \frac{763}{8} e^4 + \frac{3605}{64} e^6 + \frac{175}{32} e^8\right)$$
$$\times \left(1 - 11\, f + \frac{143}{4} f^2 - \frac{715}{16} f^3 + \frac{2431}{128} f^4\right) \sin i \sin \omega \quad ,$$

$$\rho_{91} = -\frac{1575}{8}\, e(1-e^2)^{\frac{1}{2}} \left(1 + \frac{21}{4} e^2 + \frac{35}{8} e^4 + \frac{35}{64} e^6\right)$$
$$\times \left(1 - 11\, f + \frac{143}{4} f^2 - \frac{715}{16} f^3 + \frac{2431}{128} f^4\right) \sin i \sin \omega \quad .$$

$$e_{93} = \frac{24255}{128}\, e^2(1-e^2)\left(1 + \frac{5}{4} e^2 + \frac{3}{16} e^4\right) f$$
$$\times \left(1 - \frac{39}{8} f + \frac{117}{16} f^2 - \frac{221}{64} f^3\right) \sin i \cos 3\,\omega \quad ,$$

$$\Omega_{93} = -\frac{24255}{128}\, e^3 \left(1 + \frac{5}{4} e^2 + \frac{3}{16} e^4\right)$$
$$\times \left(1 - \frac{65}{8} f + \frac{273}{16} f^2 - \frac{663}{64} f^3\right) \sin i \cos i \sin 3\omega \quad ,$$

$$\psi_{93} = -\frac{24255}{128}\, e \left(1 + \frac{27}{4} e^2 + \frac{87}{16} e^4 + \frac{5}{8} e^6\right) f$$
$$\times \left(1 - \frac{39}{8} f + \frac{117}{16} f^2 - \frac{221}{64} f^3\right) \sin i \sin 3\omega \quad ,$$

$$\rho_{93} = -\frac{40425}{32}\, e^3(1-e^2)^{\frac{1}{2}} \left(1+\frac{5}{4}e^2+\frac{3}{16}e^4\right) f$$

$$\times \left(1-\frac{39}{8}f+\frac{117}{16}f^2-\frac{221}{64}f^3\right) \sin i \sin 3\omega \quad .$$

$$e_{95} = \frac{315315}{4096}\, e^4(1-e^2) \left(1+\frac{1}{4}e^2\right) f^2 \left(1-\frac{5}{2}f+\frac{85}{56}f^2\right) \sin i \cos 5\omega \ ,$$

$$\Omega_{95} = -\frac{315315}{4096}\, e^5 \left(1+\frac{1}{4}e^2\right) f \left(1-\frac{7}{2}f+\frac{153}{56}f^2\right) \sin i \cos i \sin 5\omega \ ,$$

$$\psi_{95} = -\frac{315315}{4096}\, e^3 \left(1+\frac{11}{4}e^2+\frac{1}{2}e^4\right) f^2 \left(1-\frac{5}{2}f+\frac{85}{56}f^2\right) \sin i \sin 5\omega \ ,$$

$$\rho_{95} = -\frac{315315}{1024}\, e^5(1-e^2)^{\frac{1}{2}} \left(1+\frac{1}{4}e^2\right) f^2 \left(1-\frac{5}{2}f+\frac{85}{56}f^2\right) \sin i \sin 5\omega \quad .$$

$$e_{97} = \frac{45045}{32768}\, e^6(1-e^2)\, f^3 \left(1-\frac{17}{16}f\right) \sin i \cos 7\omega \ ,$$

$$\Omega_{97} = -\frac{45045}{32768}\, e^7 f^2 \left(1-\frac{153}{112}f\right) \sin i \cos i \sin 7\omega \ ,$$

$$\psi_{97} = -\frac{45045}{32768}\, e^5 \left(1+\frac{10}{7}e^2\right) f^3 \left(1-\frac{17}{16}f\right) \sin i \sin 7\omega \ ,$$

$$\rho_{97} = -\frac{32175}{8192}\, e^7(1-e^2)^{\frac{1}{2}} f^3 \left(1-\frac{17}{16}f\right) \sin i \sin 7\omega \quad .$$

6.4 ζ_{nk} for $n < -2$

The recurrence relations of section 6.2 and the basic equations (38) to (45) have been used to obtain the non-zero ζ_{nk}, extended for negative n as far as n = -6. Application of the results will be considered in section 8 for $n \leq -3$ and in section 9 for n = -2. The remarks of section 6.3, concerning the listing of results, apply.

<u>n = -2</u>
$$e_{-21} = -\frac{3}{2}(1-e^2)^{\frac{1}{2}} \sin i \cos \omega \ , \qquad \Omega_{-21} = \frac{3}{2}\, e(1-e^2)^{-\frac{1}{2}} \cot i \sin \omega \ ,$$

$$\psi_{-21} = \frac{3}{2}\, e^{-1}(1-e^2)^{\frac{1}{2}} \sin i \sin \omega \ , \qquad \rho_{-21} = -3e \sin i \sin \omega \quad .$$

<u>n = -3</u>
$$\Omega_{-30} = -\frac{3}{2}(1-e^2)^{-\frac{1}{2}} \left(1+\frac{3}{2}e^2\right) \cos i \ ,$$

$$\psi_{-30} = \frac{3}{2}(1-e^2)^{\frac{1}{2}} \left(1-\frac{3}{2}f\right) , \qquad \rho_{-30} = -2\left(1+\frac{3}{2}e^2\right)\left(1-\frac{3}{2}f\right) .$$

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$$e_{-32} = \frac{15}{4}\, e(1-e^2)^{\frac{1}{2}}\, f \sin 2\omega \,, \qquad \Omega_{-32} = \frac{15}{4}\, e^2(1-e^2)^{-\frac{1}{2}} \cos i \cos 2\omega ,$$

$$\psi_{-32} = \frac{15}{4}\,(1-e^2)^{\frac{1}{2}}\, f \cos 2\omega \,, \qquad \rho_{-32} = -\frac{15}{2}\, e^2 f \cos 2\omega \,.$$

<u>n = -4</u>

$$e_{-41} = \frac{15}{4}\,(1-e^2)^{\frac{1}{2}} \left(1+\frac{3}{4}e^2\right)\left(1-\frac{5}{4}f\right) \sin i \cos \omega \,,$$

$$\Omega_{-41} = -\frac{15}{4}\, e(1-e^2)^{-\frac{1}{2}} \left(1+\frac{3}{4}e^2\right)\left(1-\frac{15}{4}f\right) \cot i \sin \omega \,,$$

$$\psi_{-41} = -\frac{15}{4}\, e^{-1}(1-e^2)^{\frac{1}{2}} \left(1+\frac{9}{4}e^2\right)\left(1-\frac{5}{4}f\right) \sin i \sin \omega \,,$$

$$\rho_{-41} = \frac{45}{2}\, e \left(1+\frac{3}{4}e^2\right)\left(1-\frac{5}{4}f\right) \sin i \sin \omega \,.$$

$$e_{-43} = \frac{525}{64}\, e^2(1-e^2)^{\frac{1}{2}}\, f \sin i \cos 3\omega \,,$$

$$\Omega_{-43} = -\frac{525}{64}\, e^3(1-e^2)^{-\frac{1}{2}} \cos i \sin i \sin 3\omega \,,$$

$$\psi_{-43} = -\frac{525}{64}\, e(1-e^2)^{\frac{1}{2}}\, f \sin i \sin 3\omega \,,$$

$$\rho_{-43} = \frac{525}{32}\, e^3 f \sin i \sin 3\omega \,.$$

<u>n = -5</u>

$$\Omega_{-50} = \frac{15}{4}\,(1-e^2)^{-\frac{1}{2}} \left(1+5e^2+\frac{15}{8}e^4\right)\left(1-\frac{7}{4}f\right) \cos i \,,$$

$$\psi_{-50} = -\frac{15}{4}\,(1-e^2)^{\frac{1}{2}} \left(1+\frac{3}{4}e^2\right)\left(1-5f+\frac{35}{8}f^2\right) ,$$

$$\rho_{-50} = 3\left(1+5e^2+\frac{15}{8}e^4\right)\left(1-5f+\frac{35}{8}f^2\right) .$$

$$e_{-52} = -\frac{315}{16}\, e(1-e^2)^{\frac{1}{2}} \left(1+\frac{1}{2}e^2\right) f \left(1-\frac{7}{6}f\right) \sin 2\omega \,,$$

$$\Omega_{-52} = -\frac{315}{16}\, e^2(1-e^2)^{-\frac{1}{2}} \left(1+\frac{1}{2}e^2\right)\left(1-\frac{7}{3}f\right) \cos i \cos 2\omega \,,$$

$$\psi_{-52} = -\frac{315}{16}\,(1-e^2)^{\frac{1}{2}}\,(1+e^2)\, f \left(1-\frac{7}{6}f\right) \cos 2\omega \,,$$

$$\rho_{-52} = \frac{315}{4}\, e^2 \left(1+\frac{1}{2}e^2\right) f \left(1-\frac{7}{6}f\right) \cos 2\omega \,.$$

$$e_{-54} = -\frac{2205}{128} e^3(1-e^2)^{\frac{1}{2}} f^2 \sin 4\omega \quad ,$$

$$\Omega_{-54} = -\frac{2205}{128} e^4(1-e^2)^{-\frac{1}{2}} f \cos i \cos 4\omega \quad ,$$

$$\psi_{-54} = -\frac{2205}{128} e^2(1-e^2)^{\frac{1}{2}} f^2 \cos 4\omega \quad ,$$

$$\rho_{-54} = \frac{2205}{64} e^4 f^2 \cos 4\omega \quad .$$

<u>n = -6</u>

$$e_{-61} = -\frac{105}{16} (1-e^2)^{\frac{1}{2}} \left(1 + \frac{5}{2} e^2 + \frac{5}{8} e^4\right)\left(1 - \frac{7}{2} f + \frac{21}{8} f^2\right) \sin i \cos \omega \quad ,$$

$$\Omega_{-61} = \frac{105}{16} e(1-e^2)^{-\frac{1}{2}} \left(1 + \frac{5}{2} e^2 + \frac{5}{8} e^4\right)\left(1 - \frac{21}{2} f + \frac{105}{8} f^2\right) \cot i \sin \omega \quad ,$$

$$\psi_{-61} = \frac{105}{16} e^{-1}(1-e^2)^{\frac{1}{2}} \left(1 + \frac{15}{2} e^2 + \frac{25}{8} e^4\right)\left(1 - \frac{7}{2} f + \frac{21}{8} f^2\right) \sin i \sin \omega \quad ,$$

$$\rho_{-61} = -\frac{525}{8} e \left(1 + \frac{5}{2} e^2 + \frac{5}{8} e^4\right)\left(1 - \frac{7}{2} f + \frac{21}{8} f^2\right) \sin i \sin \omega \, .$$

$$e_{-63} = -\frac{2205}{32} e^2(1-e^2)^{\frac{1}{2}} \left(1 + \frac{3}{8} e^2\right) f \left(1 - \frac{9}{8} f\right) \sin i \cos 3\omega \quad ,$$

$$\Omega_{-63} = \frac{2205}{32} e^3(1-e^2)^{-\frac{1}{2}} \left(1 + \frac{3}{8} e^2\right)\left(1 - \frac{15}{8} f\right) \cos i \sin i \sin 3\omega \quad ,$$

$$\psi_{-63} = \frac{2205}{32} e(1-e^2)^{\frac{1}{2}} \left(1 + \frac{5}{8} e^2\right) f \left(1 - \frac{9}{8} f\right) \sin i \sin 3\omega \quad ,$$

$$\rho_{-63} = -\frac{3675}{16} e^3 \left(1 + \frac{3}{8} e^2\right) f \left(1 - \frac{9}{8} f\right) \sin i \sin 3\omega \quad .$$

$$e_{-65} = -\frac{72765}{2048} e^4(1-e^2)^{\frac{1}{2}} f^2 \sin i \cos 5\omega \quad ,$$

$$\Omega_{-65} = \frac{72765}{2048} e^5(1-e^2)^{-\frac{1}{2}} f \cos i \sin i \sin 5\omega \quad ,$$

$$\psi_{-65} = \frac{72765}{2048} e^3(1-e^2)^{\frac{1}{2}} f^2 \sin i \sin 5\omega \quad ,$$

$$\rho_{-65} = -\frac{72765}{1024} e^5 f^2 \sin i \sin 5\omega \quad .$$

7 THE ORBITAL PERIOD

The period of an orbit is perhaps its most fundamental characteristic. A good estimate of it can be made by use of equipment no more complicated than a clock. It is desirable, therefore, to consider the effect on the orbital period of the perturbations considered in this paper.

For an unperturbed orbit there is a unique period given by

$$T = \frac{2\pi}{n} \quad , \tag{59}$$

where n, the mean motion, relates to a, the semi-major axis, by

$$n^2 a^3 = \mu \quad .$$

For an orbit which is changing all the time the period must be carefully defined; different definitions lead to different quantities associated with the same orbit, though not so very different if the perturbations are small. The two most useful definitions lead to the anomalistic period, the time between successive passages through perigee, and the nodal (or draconic) period, the time between successive passages through the ascending node. We are here mainly interested in the latter, T_Ω, but a formula for the former is given at the end of the section.

Following equation (59), one would like to relate T_Ω to n_Ω, the value of n_Ω at the node; the latter does not possess a first order variation over a complete orbit, since $\Delta a = 0$. We write

$$T_\Omega = \frac{2\pi}{n_\Omega}\left(1 - \sum_{n=-\infty}^{\infty}{}' J_n \left(\frac{R}{b}\right)^n T_n\right) \quad , \tag{60}$$

where b is as defined in section 4.2 and expressions for the T_n are to be evaluated.

A good treatment to follow is that of Merson[1], who observes that, since the kinetic energy per unit mass of the satellite is $\mu\ (1/r - 1/2a)$ and the potential energy is the negative of the potential $\mu/r + U$, the equation of energy gives

$$-\frac{\mu}{2a} - U = \text{const}, \tag{61}$$

even though both a and U vary. We define a' from

$$\frac{1}{a}\left(1 + \frac{2a\ U}{\mu}\right) = \frac{1}{a'} \tag{62}$$

and then a' is constant to any - not merely first - order. If, now, n' is defined from $n'^2 a'^3 = \mu$, n' is also completely constant and is given by

$$n' = n\left(1 + \frac{3a\,U}{\mu}\right) + O(U^2) \quad . \tag{63}$$

In addition to n' and a', Merson introduces an auxiliary element φ of which the change, $\Delta\varphi$, over a complete revolution is useful. We do not need φ since we already have ρ in section 5 and $\Delta\varphi = -\Delta\rho$.

The procedure followed is to write down the first-order change over one revolution - node to node - of each side of equation (8); thus

$$\Delta M = \Delta\sigma + \int_0^{T_\Omega} n\,dt \quad . \tag{64}$$

Introducing the constant n' by means of equation (63), we have

$$\Delta M = \Delta\sigma + n'\,T_\Omega - \int_0^{T_\Omega} \frac{3\,na\,U}{\mu}\,dt \quad . \tag{65}$$

Now the change in M over a nodal revolution can be related to the changes, Δe and $\Delta\omega$, in e and ω, using eccentric and true anomaly as intermediate variables and starting from the relation $\Delta v + \Delta\omega = 2\pi$. Taking Merson's first-order terms, we have

$$\Delta M = 2\pi - (1-e^2)^{3/2}\left(\frac{r_\Omega}{p}\right)^2 \Delta\omega + (1-e^2)^{\frac{1}{2}}\,\frac{r_\Omega}{p}\left(1 + \frac{r_\Omega}{p}\right)\sin\omega\,\Delta e \quad , \quad \ldots\ldots \tag{66}$$

where

$$\frac{p}{r_\Omega} = 1 + e\cos\omega \quad .$$

The integral in equation (65) can be expressed in terms of the ρ_n of section 5; these ρ_n arose on integrating equation (7) and in fact

$$-\int_0^{T_\Omega} \frac{2}{na}\frac{\partial U}{\partial a}\,dt = \Delta\rho = 2\pi\sum_{n=-\infty}^{\infty} J_n\left(\frac{R}{b}\right)^n \rho_n \quad , \tag{67}$$

where n denotes mean motion on the L.H.S. and an indicial suffix on the R.H.S. But U_n is proportional to a^{-n-1} and so we must have

$$\int_0^{T_\Omega} \frac{3\,na\,U}{\mu}\,dt = 2\pi\sum_{n=-\infty}^{\infty}\left\{J_n\left(\frac{R}{b}\right)^n \frac{3\,\rho_n}{2(n+1)}\right\} \tag{68}$$

with the same remark about the dual use of n. We note that on the R.H.S. of the above equation $\rho_n/\overline{n+1}$ gives o/o when n = -1. Since $U_{-1} = -\mu J_{-1}/R$ we must conventionally take $\rho_{-1}/0 = -2$.

By now, from equation (65), we can express $n'T_\Omega$ in terms of the e_n, ω_n, σ_n and ρ_n; the ρ_n can be eliminated in favour of Ω_n since, by equations (42) and (43),

$$\rho_n = \sigma_n + (1-e^2)^{\frac{1}{2}} (\omega_n + \Omega_n \cos i) \quad . \tag{69}$$

To introduce the T_n, which we are trying to evaluate, we observe that from equation (63), on substituting for U_Ω from equation (13),

$$n' = n_\Omega \left(1 - \sum_{n=-\infty}^{\infty} \frac{3a\, J_n\, R^n}{r_\Omega^{n+1}} P_\ell(0)\right) \quad .$$

Hence, from equation (60),

$$n'\, T_\Omega = 2\pi \left\{ 1 - \sum_{n=-\infty}^{\infty} J_n \left(\frac{R}{b}\right)^n \left(T_n + \frac{3a\, b^n}{r_\Omega^{n+1}} P_\ell(0)\right)\right\} \quad . \tag{70}$$

Thus, combining equations (65), (66), (68) and (70), and picking out the terms involving $2\pi J_n (R/b)^n$ for each n, we have

$$\begin{aligned} T_n = \sigma_n &- \frac{3\rho_n}{2(n+1)} + (1-e^2)^{3/2} \left(\frac{r_\Omega}{p}\right)^2 \omega_n \\ &- (1-e^2)^{\frac{1}{2}} \frac{r_\Omega}{p} \left(1 + \frac{r_\Omega}{p}\right) \sin\omega\; e_n - \frac{3a\, b^n}{r_\Omega^{n+1}} P_\ell(0) \quad , \end{aligned} \tag{71}$$

where ρ_n is given by equation (69).

It is remarked that for n = -1 and n = 0 results are obtained which agree with those of section 6.1. When n = -1, taking $\rho_{-1}/0 = -2$ as earlier and $P_o(0) = 1$, we get $T_{-1} = 0$; so, neglecting other disturbing terms, $T_\Omega = 2\pi/n_\Omega$, though we note that $n_\Omega \neq n'$. When n = 0, $\sigma_o = \rho_o = -2$ and so

$$T_\Omega = \frac{2\pi}{n_\Omega}\left\{1 - J_o\left(1 - \frac{3a}{r_\Omega}\right)\right\} \;;$$

but from equation (49) of section 6.1, $\bar{a} = a_\Omega \{1 + J_o (2\bar{a}/r_\Omega - 1)\}$ and so if $\bar{n}^2\, \bar{a}^3 = \bar{\mu} = \mu\, (1 - J_o)$, $\bar{n} = n_\Omega \{1 + J_o\, (1 - 3\bar{a}/r_\Omega)\}$; hence $T_\Omega = 2\pi/\bar{n}$ as expected; we note that $\bar{n} = n'\, (1 + J_o)$ so that $n_\Omega \neq n' \neq \bar{n}$.

For completeness we give the formula for the anomalistic period, T_ω, which corresponds to equation (71). If, similar to equation (60),

$$T_\omega = \frac{2\pi}{n_\omega}\left(1 - \sum_{n=-\infty}^{\infty} J_n \left(\frac{R}{b}\right)^n T_{\omega n}\right) ,$$

then

$$T_{\omega n} = \sigma_n - \frac{3\,\rho_n}{2(n+1)} - \frac{3a\,b^n}{r_\omega^{n+1}}\, P_2\,(\sin\beta_\omega) \quad , \quad .$$

where r_ω is the perigee distance and β_ω is its latitude.

8 APPLICATION TO LUNI-SOLAR (GRAVITATIONAL) PERTURBATION THEORY

8.1 Introductory remarks

The (disturbing) gravitational field, produced by a distant body - such as the sun or moon - considered as a point mass, is axi-symmetric, the axis of symmetry being simply the line from the main centre of force - the earth - to the distant body. Hence it is to be expected that a first-order account of the effects of the disturbing field will be given by the formulae of this paper; it is only necessary to interpret the values of the J coefficients appropriately and this is done in section 8.2.

A complication arises in that the axis of symmetry of the gravitational field has hitherto been assumed to lie in the direction of the north pole. When it lies in another direction - towards the distant body - the formulae are only immediately applicable if the 'orientating' orbital elements - i, Ω and ω - are assumed to relate to this other direction (and a plane perpendicular to it) instead of to the polar axis (and equator). The formulae must be modified if i, Ω and ω are to be used in the normal way and this modification - essentially a rotation of co-ordinates - is considered in section 8.3.

In this application of the basic formulae of the paper it is first assumed that the gravitational field does not vary, i.e. that the disturbing body is stationary. If account is to be paid to the motion of the disturbing body in its own orbit relative to the main centre of force, the results for a stationary body must be averaged with respect to the body's mean anomaly. This subject is discussed in section 8.4. Yet another axis is suggested, one that is perpendicular to the orbit of the disturbing body, but the average field is not symmetric with respect to this axis unless the orbit of the disturbing body is circular.

Consideration of long-term secular perturbations, derived by a further averaging process, is postponed until section 10.3.

8.2 Interpretation of the J coefficients

The disturbing body is itself a centre of force and we suppose that the strength of its field is given by the parameter μ_d, taking the local potential to be μ_d/r_d where r_d is the distance from the body. However this is not the disturbing potential at a satellite in orbit about the main centre of force, since the main centre is itself attracted towards the disturbing centre.

Taking r and R to be the distances of the satellite and the disturbing body, respectively, from the main centre of force and introducing vector rotation based on Fig.2, we have

$$\underset{\sim}{r_d} = \underset{\sim}{r} - \underset{\sim}{R} \;;$$

the accelerations of the satellite and main centre, respectively, towards the disturbing body are then

$$-\frac{\mu_d \underset{\sim}{r}_d}{r_d^3} \qquad \text{and} \qquad \frac{\mu_d \underset{\sim}{R}}{R^3} \; .$$

It may be seen at once, on varying r, that these accelerations correspond to potential terms

$$\frac{\mu_d}{r_d} \qquad \text{and} \qquad \frac{\mu_d \; r \cos\varphi}{R^2}$$

respectively, where φ is the angle between $\underset{\sim}{r}$ and $\underset{\sim}{R}$. Hence the disturbing potential, U, is given by

$$U = \mu_d \left(\frac{1}{r_d} - \frac{r\cos\varphi}{R^2}\right) \; .$$

Now from Appendix A,

$$\frac{1}{r_d} = \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{R^{\ell+1}} P_{\ell}(\cos\varphi) = \sum_{n=-1}^{-\infty} \frac{R^n}{r^{n+1}} P_{\ell}(\cos\varphi) \;,$$

so that

$$U = \frac{\mu_d}{r} \sum \left(\frac{R}{r}\right)^n P_{\ell}(\cos\varphi) \tag{72}$$

where the summation covers n = −1 and −3, −4,.... −∞.

Hence the disturbing potential may be represented by equation (11) with the interpretation: $J_n = 0$ if $n = -2$ and $0, 1, \ldots, \infty$; $J_n = -\mu_d/\mu$ if $n = -1$ and $-3, -4, \ldots, -\infty$; R is the distance from the centre of force (earth) to the disturbing body.

The term with $n = -1$ is constant and may be disregarded as stated in section 6.1. The terms with $n = -3, -4$ etc are associated with Legendre polynomials of order 2, 3 etc in the normal way.

8.3 Perturbations caused by a stationary disturbing body

With the interpretation of the J_n just made, the perturbations of the elements are given at once by equations (36) to (45) if i, Ω and ω are taken as elements relative to the pseudo-equatorial plane for which the 'north pole' is in the direction of the disturbing body. Since we prefer to have the perturbations of the usual elements - with i, Ω and ω relative to the true equatorial plane - it is convenient to proceed by first adding dashes to i, Ω and ω when used as above, so that the perturbations given by equations (38) to (45) are to a, e, i', Ω', ω' and σ; they involve i' and ω' in their expression.

To convert to formulae involving the usual elements, relations between the dashed and undashed elements are required. They may be obtained by means of the axis transformations of Appendix F. To give the results here we make use of a system of axes Oxyz defined as follows: Ox is towards the (true) ascending node of the satellite, Oy is towards its apex and Oz is normal to the orbital plane. Since fixed axes are required, this definition must be regarded as strictly valid at one instant only, due to the variation of i and Ω. With the notation of Cook[2], we take the direction cosines of the disturbing body to be A, B and C in the Oxyz system.

Now, although i' is at once defined as the angle between the pseudo-equator and the orbital plane (the xy-plane), Ω' is not defined, since its origin is arbitrary. The transformations involve α (instead of Ω'), where α is the angle in the xy-plane from Ox (the direction of the true node) towards the pseudo-node. Then i' and ω' may be eliminated in favour of A, B and C by means of the relations

$$\omega' = \omega - \alpha \quad , \tag{73}$$

which is obvious, and

$$(-\sin i' \sin \alpha, \quad \sin i' \cos \alpha, \quad \cos i') = (A, B, C) \quad , \tag{74}$$

proved in Appendix F.

Appendix F also gives the required expressions for the perturbations Δi, $\Delta\Omega$ and $\Delta\omega$ in terms of $\Delta i'$, $\Delta\Omega'$ and $\Delta\omega'$. The expressions are:

$$\Delta i = \frac{B}{\sin i'}\,\Delta i' + A\,\Delta\Omega' \quad , \tag{75}$$

$$\sin i\,\Delta\Omega = -\frac{A}{\sin i'}\,\Delta i' + B\,\Delta\Omega' \tag{76}$$

and

$$\Delta\omega + \Delta\Omega\cos i = \Delta\omega' + \Delta\Omega'\cos i' \quad . \tag{77}$$

We now have all the relations necessary for the conversion of the dashed versions of equations (39) to (45); equation (38) is, of course, unaffected. From equation (39) we have, since $\varepsilon = 0$,

$$e_{nk} = k\,C_n^k\,e^{k-1}\,(1-e^2)^{\frac{1}{2}}\,B_h^k(e)\,A_\ell^k(i')\,\sin^k i'\,\sin k\,(\omega'-\tfrac{1}{2}\pi) \quad . \tag{78}$$

Now $A_\ell^k(i')$ is a function of $\cos i'$ and f' ($= \sin^2 i'$) so that from equation (74) we may write

$$A_\ell^k(i') = A_\ell^k\,(A^2 + B^2,\, C) \quad ,$$

and similarly $D_\ell^k(i') = D_\ell^k\,(A^2 + B^2,\, C)$, where the new function A_ℓ^k (of two arguments) is a polynomial. Also, from equation (73),

$$\cos k\,(\omega'-\tfrac{1}{2}\pi) = \cos k\,(\omega-\tfrac{1}{2}\pi)\cos k\alpha + \sin k\,(\omega-\tfrac{1}{2}\pi)\sin k\alpha \tag{79}$$

and

$$\sin k\,(\omega'-\tfrac{1}{2}\pi) = \sin k\,(\omega-\tfrac{1}{2}\pi)\cos k\alpha - \cos k\,(\omega-\tfrac{1}{2}\pi)\sin k\alpha \quad , \tag{80}$$

where, by trigonometrical expansion,

$$\cos k\alpha = \sum_{q(0<2q<k)} (-1)^q \binom{k}{2q} \cos^{k-2q}\alpha\,\sin^{2q}\alpha \tag{81}$$

and

$$\sin k\alpha = \sum_{s(0<2s+1\leq k)} (-1)^s \binom{k}{2s+1} \cos^{k-2s-1}\alpha \, \sin^{2s+1}\alpha \, . \quad (82)$$

On substituting from equations (79) to (82) in equation (78) and using equation (74), we get

$$e_{nk} = k\, C_n^k\, e^{k-1}\, (1-e^2)^{\frac{1}{2}}\, B_h^k(e)\, A_{\ell}^k\, (A^2 + B^2, C)$$

$$\times \left[\sin k\,(\omega-\tfrac{1}{2}\pi) \sum_q (-1)^q \binom{k}{2q} A^{2q} B^{k-2q} \right.$$

$$\left. + \cos k\,(\omega-\tfrac{1}{2}\pi) \sum_s (-1)^s \binom{k}{2s+1} A^{2s+1} B^{k-2s-1} \right] \, . \quad (83)$$

For i and Ω we have first, from equations (40) and (41),

$$\frac{i'_{nk}}{\sin i'} = -\,k\, C_n^k\, e^k (1-e^2)^{-\frac{1}{2}}\, B_h^k(e)\, \frac{\cos i'\, A_{\ell}^k(i')}{f'} \sin^k i' \, \sin k\,(\omega'-\tfrac{1}{2}\pi)$$

and

$$\Omega'_{nk} = C_n^k\, e^k (1-e^2)^{-\frac{1}{2}}\, B_h^k(e)\, \frac{D_{\ell}^k(i')}{f'} \sin^k i' \, \cos k\,(\omega'-\tfrac{1}{2}\pi) \, .$$

Applying equations (75), (76) and (79) to (82), these lead to

$$i_{nk} = - C_n^k\, e^k (1-e^2)^{-\frac{1}{2}}\, B_h^k(e)$$

$$\times \Bigg[\frac{\sin k\,(\omega-\frac{1}{2}\pi)}{A^2 + B^2} \Big\{ k\, A_\ell^k(A^2 + B^2, C) \sum_q (-1)^q \binom{k}{2q} A^{2q} B^{k-2q+1}\, C$$

$$+ D_\ell^k\, (A^2 + B^2, C) \sum_s (-1)^s \binom{k}{2s+1} A^{2s+2} B^{k-2s-1} \Big\}$$

$$+ \frac{\cos k\,(\omega-\frac{1}{2}\pi)}{A^2 + B^2} \Big\{ k\, A_\ell^k(A^2 + B^2, C) \sum_s (-1)^s \binom{k}{2s+1} A^{2s+1} B^{k-2s}\, C$$

$$- D_\ell^k(A^2 + B^2, C) \sum_q (-1)^q \binom{k}{2q} A^{2q+1} B^{k-2q} \Big\} \Bigg]$$

$$\dots\dots \quad (84)$$

and

$$\Omega_{nk} \sin i = C_n^k\, e^k\, (1-e^2)^{-\frac{1}{2}}\, B_h^k(e)$$

$$\times \Bigg[\frac{\sin k\,(\omega-\frac{1}{2}\pi)}{A^2 + B^2} \Big\{ k\, A_\ell^k(A^2 + B^2, C) \sum_q (-1)^q \binom{k}{2q} A^{2q+1} B^{k-2q}\, C$$

$$- D_\ell^k(A^2 + B^2, C) \sum_s (-1)^s \binom{k}{2s+1} A^{2s+1} B^{k-2q} \Big\}$$

$$+ \frac{\cos k\,(\omega-\frac{1}{2}\pi)}{A^2 + B^2} \Big\{ k\, A_\ell^k(A^2 + B^2, C) \sum_s (-1)^s \binom{k}{2s+1} A^{2s+2} B^{k-2s-1} C$$

$$+ D_\ell^k(A^2 + B^2, C) \sum_q (-1)^q \binom{k}{2q} A^{2q} B^{k-2q+1} \Big\} \Bigg].$$

$$\dots\dots \quad (85)$$

For the remaining two elements, ω and σ, we avoid highly cumbersome expressions by using – as in section 5 – the auxiliary quantities ψ and ρ. From equation (77),

$$\Delta\psi = \Delta\psi'$$

and so the usual analysis, starting from equations (44) and (45), leads to

$$(\omega_{nk} + \Omega_{nk} \cos i =)\ \psi_{nk} = C_n^k\, e^{k-2}\, (1-e^2)^{\frac{1}{2}}\, E_n^k(e)\, \Lambda_\ell^k(A^2 + B^2, C)$$

$$\times \left[\cos k\,(\omega - \tfrac{1}{2}\pi) \sum_q (-1)^q \binom{k}{2q} A^{2q}\, B^{k-2q} \right.$$

$$\left. - \sin k\,(\omega - \tfrac{1}{2}\pi) \sum_s (-1)^s \binom{k}{2s+1} A^{2s+1}\, B^{k-2s-1} \right] \qquad \ldots\ (86)$$

and

$$(\sigma_{nk} + (1-e^2)^{\frac{1}{2}}\, \psi_{nk} =)\ \rho_{nk} = 2(n+1)\, C_n^k\, e^k\, B_h^k(e)\, \Lambda_\ell^k(A^2 + B^2, C)$$

$$\times \left[\cos k\,(\omega - \tfrac{1}{2}\pi) \sum_q (-1)^q \binom{k}{2q} A^{2q}\, B^{k-2q} \right.$$

$$\left. - \sin k\,(\omega - \tfrac{1}{2}\pi) \sum_s (-1)^s \binom{k}{2s+1} A^{2s+1}\, B^{k-2s-1} \right]. \qquad \ldots\ (87)$$

For use in the next section, we also give the formula for the potential component U_{nk}, for $n < -3$. From equation (35), writing $J_n = -\mu_d/\mu$,

$$U_{nk}(av) = -\mu_d\, C_n^k\, a^{-1-n}\, R^n\, B_h^k(e)\, e^k\, \Lambda_\ell^k(A^2 + B^2, C)$$

$$\times \left[\cos k\,(\omega - \tfrac{1}{2}\pi) \sum_q (-1)^q \binom{k}{2q} A^{2q}\, B^{k-2q} \right.$$

$$\left. - \sin k\,(\omega - \tfrac{1}{2}\pi) \sum_s (-1)^s \binom{k}{2s+1} A^{2s+1}\, B^{k-2s-1} \right]. \qquad (88)$$

When $n = -3$, equations (83) to (86) lead at once to the formulae of Cook[2], on multiplying by the appropriate factor from equation (36), viz $-2\pi\ (a/R)^3\ \mu_d/\mu$, and replacing μ_d/R^3 by Cook's constant K and a^3 by μ/n^2, where n is the mean motion. Ref. 2 does not give the formula for $\Delta\sigma$, but this may be obtained from equation (87). For completeness we list all six formulae ($\Delta = \Delta_{-3}$), as follows:-

$$\Delta a = 0 \ , \tag{89}$$

$$\Delta e = -\frac{15\pi \mu_d e (1-e^2)^{\frac{1}{2}}}{R^3 n^2}\left[AB\cos 2\omega - \frac{1}{2}(A^2 - B^2)\sin 2\omega\right]; \qquad \ldots\ldots (90)$$

$$\Delta i = \frac{3\pi \mu_d (1-e^2)^{-\frac{1}{2}} C}{2 R^3 n^2}\left[A(2+3e^2 + 5e^2 \cos 2\omega) + 5 B e^2 \sin 2\omega\right] , \tag{91}$$

$$\Delta\Omega = \frac{3\pi \mu_d (1-e^2)^{-\frac{1}{2}} C}{2 R^3 n^2 \sin i}\left[5 A e^2 \sin 2\omega + B(2 + 3e^2 - 5 e^2 \cos 2\omega)\right] , \qquad \ldots\ldots (92)$$

$$\Delta\omega + \cos i \, \Delta\Omega = \frac{3\pi \mu_d (1-e^2)^{\frac{1}{2}}}{2 R^3 n^2}\left[5\{2AB\sin 2\omega + (A^2 - B^2)\cos 2\omega\} - \{2 - 3(A^2 + B^2)\}\right] \qquad \ldots\ldots (93)$$

and

$$\Delta\sigma + (1-e^2)^{\frac{1}{2}}(\Delta\omega + \cos i \, \Delta\Omega) = \frac{\pi \mu_d}{R^3 n^2}\left[(2+3e^2)\{2-3(A^2+B^2)\} - 15 e^2 \{2AB\sin 2\omega + (A^2 - B^2)\cos 2\omega\}\right] . \qquad \ldots\ldots (94)$$

The author has written a Pegasus computer sub-routine based on formulae (89) to (94). One point which is worth noting is that the sub-routine takes these formulae exactly as they stand and does not include the 'second-order' terms ($n = -4$) for $\Delta\omega$ which Cook adds to equation (93). These terms are, in fact, no more important for the element ω than for any other element. They appear more important because of a factor e in the denominator, occurring for $\Delta\omega$ only; but this merely reflects the indeterminacy of ω when e is small. If the elements e and ω are replaced by elements $e \sin \omega$ and $e \cos \omega$, then

'second order' terms for the new elements are of the same order of magnitude as similar terms for the remaining elements. This must be true since

$$\left.\begin{aligned} \Delta(e \sin\omega) &= \sin\omega\,\Delta e + e\cos\omega\,\Delta\omega \\ \text{and} \qquad & \\ \Delta(e\cos\omega) &= \cos\omega\,\Delta e - e\sin\omega\,\Delta\omega \quad . \end{aligned}\right\} \tag{95}$$

Indeed, if e is small and ω large, all perturbations to e and ω must be incorporated indirectly, instead of by direct addition to e and ω. Using $e \sin\omega$ and $e\cos\omega$ as intermediate elements, the procedure is (i) to evaluate perturbations to these elements, given Δe and $\Delta\omega$, (ii) to add the perturbations to $e\sin\omega$ and $e\cos\omega$ and (iii) to calculate the new e and ω.

When $n = -4$, equations (83) to (87) lead to formulae which are listed below ($\Delta = \Delta_{-4}$). The first five of formulae (96) to (101) agree with results obtained by Allan (end of section 4.2 in Ref.24), who uses a notation by which they can be expressed more compactly. Allan, like Cook, does not give an expression for $\Delta\sigma$. Our results are:-

$$\Delta a = 0 \quad , \tag{96}$$

$$\Delta e = \frac{15\pi\,\mu_d\, a(1-e^2)^{\frac{1}{2}}}{32\,R^4\,n^2}\left[(4+3e^2)(4-5A^2-5B^2)(A\sin\omega - B\cos\omega) + 35e^2\left\{B\cos3\omega\,(3A^2-B^2) + A\sin3\omega\,(3B^2-A^2)\right\}\right] , \tag{97}$$

$$\Delta i = -\frac{15\pi\,\mu_d\, ae(1-e^2)^{-\frac{1}{2}}C}{32\,R^4\,n^2}\left[(4+3e^2)\left\{10AB\sin\omega + (15A^2+5B^2-4)\cos\omega\right\} + 35e^2\left\{2AB\sin3\omega + (A^2-B^2)\cos3\omega\right\}\right] , \tag{98}$$

$$\Delta\Omega = -\frac{15\pi\,\mu_d\, ae(1-e^2)^{-\frac{1}{2}}C}{32\,R^4\,n^2\sin i}\left[(4+3e^2)\left\{10AB\cos\omega + (5A^2+15B^2-4)\sin\omega\right\} - 35e^2\left\{2AB\cos3\omega - (A^2-B^2)\sin3\omega\right\}\right] , \tag{99}$$

$$\Delta\omega + \cos i\,\Delta\Omega = \frac{15\pi\,\mu_d\,ae^{-1}(1-e^2)^{\frac{1}{2}}}{32\,R^4\,n^2} \times \left[(4+9e^2)(4-5A^2-5B^2)(A\cos\omega + B\sin\omega) - 35e^2\left\{B\sin 3\omega\,(3A^2-B^2) - A\cos 3\omega(3B^2-A^2)\right\}\right] \qquad \text{.... (100)}$$

and

$$\Delta\sigma + (1-e^2)^{\frac{1}{2}}\,(\Delta\omega + \cos i\,\Delta\Omega) = \frac{15\pi\,\mu_d\,ae}{16\,R^4\,n^2} \times \left[3(4+3e^2)(5A^2+5B^2-4)(A\cos\omega + B\sin\omega) + 35e^2\left\{B\sin 3\omega\,(3A^2-B^2) - A\cos 3\omega(3B^2-A^2)\right\}\right] \qquad \text{.... (101)}$$

When n = -5, equations (83) to (87) lead to quite complicated expressions. If these expressions are expanded in powers of e, however, the leading terms are fairly simple and may be compared with results obtained by Smith[25]. Ref.25, like Refs.2 and 24, does not give $\Delta\sigma$, but we get this from equation (87). Following the point made earlier, in connection with equation (93) and Ref.2, we note that since Ref.25 omits e terms for Δe, all terms should be omitted for $\Delta\omega$, the effective error being then only $O(e)$. On this basis we list the Δ_{-5} formulae for all six elements as follows:-

$$\Delta a = 0\ , \qquad (102)$$

$$\Delta e = O(e)\ , \qquad (103)$$

$$\Delta i = -\frac{15\pi\,\mu_d\,a^2}{2R^5\,n^2}\,AC\left\{1-\frac{7}{4}\,(A^2+B^2)\right\} + O(e^2)\ , \qquad (104)$$

$$\Delta\Omega = -\frac{15\pi\,\mu_d\,a^2}{2\,R^5\,n^2\,\sin i}\,BC\left\{1-\frac{7}{4}\,(A^2+B^2)\right\} + O(e^2)\ , \qquad (105)$$

$$\Delta\omega + \cos i\,\Delta\Omega = O(1) \qquad (106)$$

and

$$\Delta\sigma + (1-e^2)^{\frac{1}{2}}\,(\Delta\omega + \cos i\Delta\Omega) = -\frac{6\pi\,\mu_d\,a^2}{R^5\,n^2}\left\{1-5(A^2+B^2) + \frac{35}{8}\,(A^2+B^2)^2\right\} + O(e^2)\ . \qquad \text{.... (107)}$$

Equations (102) to (107) agree with Smith's results though there is an important difference in notation: the direction cosines A,B,C are denoted by C,A,B respectively in Ref.25. It is important to note that there is no contradiction in giving, in equation (107), terms of an order which are neglected in equation (106) : $\Delta\sigma + \Delta\omega$, when e is small, has to be known more accurately than $\Delta\omega$. The leading terms for $\Delta\omega$, which we have argued should not be listed in equation (106), are given by Smith. However his expression contains some errors which he has confirmed in a private communication. On working out the leading terms for both e and ω, we get, instead of (103) and (106),

$$\Delta e = \frac{315\pi\, \mu_d\, a^2 e}{8\, R^5\, n^2}\left\{1 - \frac{7}{6}(A^2+B^2)\right\}\{2\, AB \cos 2\omega + (B^2-A^2)\sin 2\omega\} + O(e^3)$$

and

$$\Delta\omega + \cos i\, \Delta\Omega = \frac{15\pi\, \mu_d\, a^2}{2\, R^5\, n^2}$$

$$\times\left[\left\{1-5(A^2+B^2) + \frac{35}{8}(A^2+B^2)^2\right\} - \frac{21}{4}\left\{1 - \frac{7}{6}(A^2+B^2)\right\}\{2\, AB \sin 2\omega + (A^2-B^2)\cos 2\omega\}\right] + O(e^2) \quad .$$

The results given in this section, for n = -3, -4 and -5, could be obtained directly by applying equations (73) to (77) to the explicit expressions listed in section 6.4, instead of by appeal to the general equations (83) to (87).

8.4 Perturbations caused by a disturbing body in a Kepler orbit

In allowing for the motion of the disturbing body - with orbit assumed to be a fixed Kepler ellipse around the earth - we are for the first time considering a potential field which varies with time. Over an integral number of revolutions of the disturbing body the difficulty may be overcome by averaging with respect to the mean anomaly of the body, i.e. with respect to time. If the Kepler ellipse has zero eccentricity the average field is then axi-symmetric. In general, however, it is not axi-symmetric, though the axis of the orbital plane of the disturbing body is still a special direction. The average field may be thought of as due to an 'ellipse of mass' in the orbital plane of the body.

The analysis is carried out in two parts. In the first, advantage is taken of the special axis even when it is not an axis of symmetry - the

disturbing body is not restricted to a circular orbit. As in section 8.3, special elements may be introduced - this time we use double dashes - such that i'', Ω'' and ω'' relate to the orbit of the disturbing body as 'equator' while the special axis points to the 'north pole'.

Now general formulae for all the elements have been given by equations (83) to (87) of section 8.3. It would be perfectly reasonable to use these, introducing the double dash notation and interpreting A,B and C appropriately. However, an averaging procedure has to be carried out and it is simplest just to perform this on the potential $U_{nk}(av)$ given by equation (88). Lagrange's equations may then be set up in terms of the new $U_{nk}(av, av)$ and the R.H.S. of each is constant, as in section 5.

In the second stage of the analysis it is convenient to use a new notation. Although the $\Delta\zeta$ notation, for each element ζ, was convenient for the change in ζ when $d\zeta/dt$ was integrated over a revolution of the satellite, now that a second integration is involved it is preferable to give expressions for the average rates of change of the ζ's relative to a complete revolution of both the satellite and the disturbing body. The derivative notation is appropriate, even though the original equations have been doubly integrated; so the average rate of change of ζ is denoted by $d\bar{\zeta}/dt$.

Writing $d\bar{\zeta}/dt = \sum_{n,k} d\,\bar{\zeta}_{nk}/dt$, the Lagrangian equations formed in the first stage of the analysis give $d\,\bar{\zeta}_{nk}/dt$, with ζ = a, e, i'', Ω'', ω'' and σ, in terms of a, e, i'', Ω'' and ω''. The second stage consists in the replacement, by means of axis transformations, of i'', Ω'' and ω'' by the usual elements i, Ω and ω.

We start, then, by averaging $U_{nk}(av)$ with respect to the mean anomaly of the disturbing body.

Let the axis system $Ox''y''z''$ be similar to the system Oxyz of section 8.3, except that Ox'' is now towards the pseudo-node given by the orbit of the disturbing body. Define the system $OX''Y''Z$ by this orbit, such that OX'' coincides with Ox''. Let a_d, e_d, ω_d, v_d and M_d refer to the disturbing body and its orbit, and let θ_d be the angle to the perigee measured from OX''. It is clear, from Fig.4 for example, that $\theta_d = \omega_d - \Omega''$, measuring Ω'' from the (fixed) node of the disturbing body. Then if P is the instantaneous position of the disturbing body, the direction cosines of OP, in the system $Ox''y''z''$, are given by

$$(A,B,C) = (\cos u_d,\ \cos i'' \sin u_d,\ -\sin i'' \sin u_d)\ , \qquad (108)$$

as may be seen from Fig. 5, u_d being $(\theta_d + v_d)$ and i" being the angle by which the x"y" plane is inclined to the X"Y" plane.

Hence from equation (88), with ω'' in place of ω

$U_{nk}(av)$

$$= - \mu_d C_n^k a^{-1-n} R^n B_h^k(e) e^k A_\ell^k (\cos^2 u_d + \cos^2 i'' \sin^2 u_d, - \sin i'' \sin u_d)$$

$$\Big[\cos k (\omega''-\tfrac{1}{2}\pi) \sum_q (-1)^q \binom{k}{2q} \cos^{2q} u_d \cos^{k-2q} i'' \sin^{k-2q} u_d$$

$$- \sin k (\omega''-\tfrac{1}{2}\pi) \sum_s (-1)^s \binom{k}{2s+1} \cos^{2s+1} u_d \cos^{k-2s-1} i'' \sin^{k-2s-1} u_d \Big] \quad .... \quad (109)$$

$= - \mu_d C_n^k a^{-n-1} R^n B_h^k(e) e^k F(v_d)$, say, where it is only necessary to show one argument of this F, since $u_d = \theta_d + v_d$ and other quantities are constant. Then, writing

$$R = \frac{a_d (1-e_d^2)}{1+e_d \cos v_d}$$

and, as in section 4.2,

$$d M_d = \frac{R^2 \, dv_d}{a_d^2 (1-e_d^2)^{\frac{1}{2}}} ,$$

it follows at once that

$$U_{nk}(av, av) = - \mu_d C_n^k a^{-1-n} e^k B_h^k(e) a_d^n (1-e_d^2)^{n+1\frac{1}{2}}$$
$$\times \frac{1}{2\pi} \int_0^{2\pi} F(v_d)(1+e \cos v_d)^{-n-2} dv_d \quad . \quad (110)$$

Equation (110) can be developed analytically, on the lines of section 4.2. However, the general development becomes complex and we do not embark upon it here. Instead, we use equation (110) to give expressions for particular cases of n and k, performing each integration separately. The algebra is still somewhat tedious, but the majority of the terms disappear on integration. The particular cases chosen are essentially those which received special attention

in section 8.3, viz $n = -3$, for both $k = 0$ and $k = 2$; $n = -4$, for both $k = 1$ and $k = 3$, and $n = -5$, for $k = 0$ only (we differ here from section 8.3 in giving all terms instead of leading terms only).

For $n = -3$ and $k = 0$,

$$F(v_d) = 1 - \frac{3}{2} (\cos^2 u_d + \cos^2 i'' \sin^2 u_d), \qquad \text{where } u_d = \theta_d + v_d ,$$

leading to

$$U_{-3,0}(av,\ av) = - \frac{\mu_d \ (1-e_d^2)^{-3/2}}{8\ a_d^3} \ a^2 \left(1 + \frac{3}{2} e^2\right) (1-3 \cos^2 i'') \ . \qquad \ldots\ (111)$$

For $n = -3$ and $k = 2$,

$$F(v_d) = \cos 2\omega'' (\cos^2 u_d - \cos^2 i'' \sin^2 u_d) + 2 \sin 2\omega'' \cos u_d \cos i'' \sin u_d$$

leading to

$$U_{-3,2}(av,\ av) = \frac{15\mu_d \ (1-e_d^2)^{-3/2}}{16\ a_d^3} \ a^2 e^2 \sin^2 i'' \cos 2\omega'' \ . \qquad (112)$$

For $n = -4$ and $k = 1$,

$$F(v_d) = \left\{1 - \frac{5}{4} (\cos^2 u_d + \cos^2 i'' \sin^2 u_d)\right\} \times (\cos u_d \cos \omega'' + \cos i'' \sin u_d \sin \omega'') ,$$

leading to

$$U_{-4,1}(av,\ av) = - \frac{15\mu_d \ e_d (1-e_d^2)^{-5/2}}{16\ a_d^4} \ a^3 e \left(1 + \frac{3}{4} e^2\right) \times \left\{\left(1 - \frac{5}{4} \sin^2 i''\right) \cos \theta_d \cos \omega'' + \left(1 - \frac{15}{4} \sin^2 i''\right) \cos i'' \sin \theta_d \sin \omega''\right\} . \qquad \ldots\ (113)$$

For $n = -4$ and $k = 3$,

$$F(v_d) = \sin 3\omega'' \, (3 \cos^2 u_d \cos i'' \sin u_d - \cos^3 i'' \sin^3 u_d) + \cos 3\omega'' \, (\cos^3 u_d - 3 \cos u_d \cos^2 i'' \sin^2 u_d) \; ,$$

leading to

$$U_{-4,3}(\text{av, av}) = -\frac{525\mu_d \, e_d (1-e_d^2)^{-5/2}}{256 \, a_d^4} \, a^3 e^3 \sin^2 i'' \, (\cos \theta_d \cos 3\omega'' + \cos i'' \sin \theta_d \sin 3\omega'') \, . \quad \ldots\ldots \; (114)$$

For $n = -5$ and $k = 0$,

$$F(v_d) = 1 - 5 \, (\cos^2 u_d + \cos^2 i'' \sin^2 u_d) + \frac{35}{8} (\cos^2 u_d + \cos^2 i'' \sin^2 u_d)^2$$

leading to

$$U_{-5,0}(\text{av, av}) = \frac{9\mu_d \, (1-e_d^2)^{-7/2}}{64 \, a_d^5} \, a^4 \left(1 + 5 \, e^2 + \frac{15}{8} e^4\right) \left\{\left(1 + \frac{3}{2} e_d^2\right) \left(1 - 5 \sin^2 i'' + \frac{35}{8} \sin^4 i''\right) + \frac{15}{4} e_d^2 \sin^2 i'' \cos 2\theta_d \left(1 - \frac{7}{6} \sin^2 i''\right)\right\} . \quad \ldots\ldots \; (115)$$

For each of these cases, the first part of the analysis may be completed by writing down Lagrange's planetary equations in terms of $U_{nk}(\text{av, av})$. These are, in effect, already solved, since every term on each R.H.S. is constant. But in the second part of the analysis the usual elements are to be restored. Since we aim at listing formulae for $d\,\bar{\zeta}_{nk}/dt$ at the end of the section, it is convenient at this point to obtain $d\,\bar{i}''_{nk}/dt$ and $d\,\Omega''_{nk}/dt$ only. For the

former, $\partial U(av, av)/\partial \Omega''$ is involved, so that it is necessary to replace θ_d by $\omega_d - \Omega''$ before the partial differentiation.

For n = -3 and k = 0, equations (4), (5) and (111) lead to

$$\frac{d\, \bar{i}''_{-3,0}}{dt} = 0 \tag{116}$$

and

$$\frac{d\, \bar{\Omega}''_{-3,0}}{dt} = -\frac{3\mu_d (1-e_d^2)^{-3/2}}{4\, a_d^3} \frac{(1-e^2)^{-\frac{1}{2}} (1 + 1\frac{1}{2} e^2)}{n} \cos i'' \;. \tag{117}$$

For n = -3 and k = 2, equations (4), (5) and (112) lead to

$$\frac{d\, \bar{i}''_{-3,2}}{dt} = -\frac{15\mu_d (1-e_d^2)^{-3/2}}{8\, a_d^3} \frac{e^2 (1-e^2)^{-\frac{1}{2}}}{n} \cos i'' \sin i'' \sin 2\,\omega'' \tag{118}$$

and

$$\frac{d\, \bar{\Omega}''_{-3,2}}{dt} = \frac{15\mu_d (1-e_d^2)^{-3/2}}{8\, a_d^3} \frac{e^2 (1-e^2)^{-\frac{1}{2}}}{n} \cos i'' \cos 2\,\omega'' \;. \tag{119}$$

For n = -4 and k = 1, equations (4), (5) and (113) lead to

$$\frac{d\, \bar{i}''_{-4,1}}{dt} = \frac{15\mu_d\, e_d (1-e_d^2)^{-5/2}}{32\, a_d^4} \frac{ae(1-e^2)^{-\frac{1}{2}} (1 + \frac{3}{4} e^2)}{n} \sin i''$$

$$\times \left\{ 5 \cos i'' \cos \theta_d \sin \omega'' + \left(7 - \frac{15}{2} \sin^2 i''\right) \sin \theta_d \cos \omega'' \right\} \tag{120}$$

and

$$\frac{d\, \bar{\Omega}''_{-4,1}}{dt} = \frac{15\mu_d\, e_d (1-e_d^2)^{-5/2}}{32\, a_d^4} \frac{ae(1-e^2)^{-\frac{1}{2}} (1 + \frac{3}{4} e^2)}{n}$$

$$\times \left\{ 5 \cos i'' \cos \theta_d \cos \omega'' + \left(17 - \frac{45}{2} \sin^2 i''\right) \sin \theta_d \sin \omega'' \right\} . \tag{121}$$

For $n = -4$ and $k = 3$, equations (4), (5) and (114) lead to

$$\frac{d\,\bar{i}''_{-4,3}}{dt} = -\frac{525\mu_d\; e_d(1-e_d^2)^{-5/2}}{256\; a_d^4}\,\frac{ae^3(1-e^2)^{-\frac{1}{2}}\sin i''}{n}$$

$$\times\left\{(2-3\sin^2 i'')\sin\theta_d\cos 3\,\omega'' - 2\cos i''\cos\theta_d\sin 3\,\omega''\right\} \qquad \ldots\ldots (122)$$

and

$$\frac{d\,\bar{\Omega}''_{-4,3}}{dt} = -\frac{525\mu_d\; e_d(1-e_d^2)^{-5/2}}{256\; a_d^4}\,\frac{ae^3(1-e^2)^{-\frac{1}{2}}}{n}$$

$$\times\left\{2\cos i''\cos\theta_d\cos 3\,\omega'' + (2-3\sin^2 i'')\sin\theta_d\sin 3\,\omega''\right\}. \qquad \ldots\ldots (123)$$

For $n = -5$ and $k = 0$, equations (4), (5) and (115) lead to

$$\frac{d\,\bar{i}_{-5,0}}{dt} = -\frac{135\mu_d\; e_d^2(1-e_d^2)^{-7/2}}{128\; a_d^5}\,\frac{a^2(1-e^2)^{-\frac{1}{2}}\left(1+5e^2+\frac{15}{8}e^4\right)}{n}$$

$$\times\sin i''\sin 2\theta_d\left(1-\frac{7}{6}\sin^2 i''\right) \qquad \ldots\ldots (124)$$

and

$$\frac{d\,\bar{\Omega}''_{-5,0}}{dt} = -\frac{45\mu_d\;(1-e_d^2)^{-7/2}}{32\; a_d^5}\,\frac{a^2(1-e^2)^{-\frac{1}{2}}\left(1+5e^2+\frac{15}{8}e^4\right)}{n}\cos i''$$

$$\times\left\{\left(1+\frac{3}{2}e_d^2\right)\left(1-\frac{7}{4}\sin^2 i''\right)-\frac{3}{4}e_d^2\cos 2\theta_d\left(1-\frac{7}{3}\sin^2 i''\right)\right\}. \qquad \ldots\ldots (125)$$

We now give the relations by means of which i'', Ω'' and ω'' may be eliminated in favour of the usual i, Ω and ω. Let i_d and Ω_d, like a_d etc, denote elements of the disturbing body's orbit; they are referred, like the elements i and Ω

of the satellite orbit, to the standard (equator-equinox based) axes. Let α be, as in section 8.3, the angle in the orbital plane of the satellite, from the true node to the pseudo-node, i.e., to the intersection with the orbital plane of the disturbing body.

Then it is shown in Appendix G that $d\bar{i}/dt$, $d\bar{\Omega}/dt$ and $d\bar{\omega}/dt$ are given in terms of the double dashed derivatives by

$$\frac{d\bar{i}}{dt} = \cos\alpha \frac{d\bar{i}''}{dt} - \sin\alpha \sin i'' \frac{d\bar{\Omega}''}{dt} , \tag{126}$$

$$\sin i \frac{d\bar{\Omega}}{dt} = \sin\alpha \frac{d\bar{i}''}{dt} + \cos\alpha \sin i'' \frac{d\bar{\Omega}''}{dt} \tag{127}$$

and

$$\frac{d\bar{\omega}}{dt} + \cos i \frac{d\bar{\Omega}}{dt} = \frac{d\bar{\omega}''}{dt} + \cos i'' \frac{d\bar{\Omega}''}{dt} . \tag{128}$$

Appendix G also establishes the important transformation equations which relate the axis system Oxyz of section 8.3 to the corresponding system $Ox_dy_dz_d$ for the disturbing body, where Ox_d points towards its node. In matrix form the equations may be combined as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{pmatrix} \begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix} , \tag{129}$$

where

$$L_1 = \cos(\Omega-\Omega_d) , \quad M_1 = \cos i_d \sin(\Omega-\Omega_d) , \quad N_1 = -\sin i_d \sin(\Omega-\Omega_d) ,$$

$$L_2 = -\cos i \sin(\Omega-\Omega_d) , \quad M_2 = \sin i_d \sin i + \cos i_d \cos i \cos(\Omega-\Omega_d) ,$$

$$N_2 = \cos i_d \sin i - \sin i_d \cos i \cos(\Omega-\Omega_d) , \quad L_3 = \sin i \sin(\Omega-\Omega_d) ,$$

$$M_3 = \sin i_d \cos i - \cos i_d \sin i \cos(\Omega-\Omega_d)$$

and

$$N_3 = \cos i_d \cos i + \sin i_d \sin i \cos(\Omega-\Omega_d) .$$

Some of the elements of the transformation matrix were introduced by Smith[25]. Indeed L_1, L_2, L_3, M_1, M_2 and M_3 are respectively Smith's Y_3, Y_1, Y_2, X_3, X_1 and X_2, the permutation of suffixes being in conformity with the permutation of A,B and C as used in Ref.25.

From these transformation equations Appendix G derives, finally, the relations by means of which substitutions may be made for i", ω", α and θ_d. These are

$$\omega'' = \omega - \alpha \quad , \tag{130}$$

$$\sin i'' \sin \alpha = - N_1 \quad , \tag{131}$$

$$\sin i'' \cos \alpha = N_2 \quad , \tag{132}$$

$$\cos i'' = N_3 \quad , \tag{133}$$

$$\sin i'' \sin \theta_d = - L_3 \cos \omega_d - M_3 \sin \omega_d \tag{134}$$

and

$$\sin i'' \cos \theta_d = L_3 \sin \omega_d - M_3 \cos \omega_d \quad . \tag{135}$$

The way is now clear to giving the final expressions for the $d\,\bar{\xi}_{nk}/dt$, for the particular cases chosen, introducing the remaining Lagrangian equations (2), (3), (6) and (7). Every $d\,\bar{a}_{nk}/dt$ is zero and may be forgotten. For $d\,\bar{i}_{nk}/dt$ and $d\,\bar{\Omega}_{nk}/dt$, $d\bar{i}''_{nk}/dt$ and $d\bar{\Omega}''_{nk}/dt$ in equations (116) to (125) are used through equations (126) and (127). Equation (128) may be used in the standard way, each side being $d\,\bar{\Psi}/dt$, in the notation of section 5. The quantities L_1, ..., N_3 of equation (129) are used in the final expressions since these quantities depend only on the orbital parameters of the satellite and the disturbing body.

For $n = -3$ and $k = 0$,

$$\frac{d\,\bar{e}_{-3,0}}{dt} = 0 \quad , \tag{136}$$

$$\frac{d\,\bar{i}_{-3,0}}{dt} = - \frac{3\mu_d\,(1-e_d^2)^{-3/2}}{4\,a_d^3\,n}\,(1-e^2)^{-\frac{1}{2}}\left(1+\frac{3}{2}e^2\right)N_1\,N_3 \quad , \quad \ldots\ldots \tag{137}$$

$$\frac{d\,\bar{\Omega}_{-3,0}}{dt} = -\frac{3\mu_d\,(1-e_d^2)^{-3/2}}{4\,a_d^3\,n\,\sin i}(1-e^2)^{-\frac{1}{2}}\left(1+\frac{3}{2}e^2\right)N_2\,N_3\ , \qquad \ldots\ (138)$$

$$\frac{d\,\bar{\omega}_{-3,0}}{dt} + \cos i\,\frac{d\,\bar{\Omega}_{-3,0}}{dt} = -\frac{3\mu_d\,(1-e_d^2)^{-3/2}}{8\,a_d^3\,n}(1-e^2)^{\frac{1}{2}}\,(1-3\,N_3^2) \qquad (139)$$

and

$$\frac{d\,\bar{\sigma}_{-3,0}}{dt} = \frac{\mu_d\,(1-e_d^2)^{-3/2}}{8\,a_d^3\,n}(7+3\,e^2)\,(1-3\,N_3^2)\ . \qquad (140)$$

For $n = -3$ and $k = 2$,

$$\frac{d\,\bar{e}_{-3,2}}{dt} = \frac{15\mu_d\,(1-e_d^2)^{-3/2}}{8\,a_d^3\,n}e(1-e^2)^{\frac{1}{2}}\,\{(N_2^2-N_1^2)\,\sin 2\omega + 2\,N_1\,N_2\,\cos 2\omega\}\ , \qquad (141)$$

$$\frac{d\,\bar{i}_{-3,2}}{dt} = -\frac{15\mu_d(1-e_d^2)^{-3/2}}{8\,a_d^3\,n}e^2(1-e^2)^{-\frac{1}{2}}\,N_3\,\{N_1\,\cos 2\omega + N_2\sin 2\omega\}\ , \qquad \ldots\ (142)$$

$$\frac{d\,\bar{\Omega}_{-3,2}}{dt} = \frac{15\mu_d\,(1-e_d^2)^{-3/2}}{8\,a_d^3\,n\,\sin i}e^2(1-e^2)^{-\frac{1}{2}}\,N_3\,(N_2\cos 2\omega - N_1\,\sin 2\omega)\ , \qquad \ldots\ (143)$$

$$\frac{d\,\bar{\omega}_{-3,2}}{dt} + \cos i\,\frac{d\,\bar{\Omega}_{-3,2}}{dt} = \frac{15\mu_d\,(1-e_d^2)^{-3/2}}{8\,a_d^3\,n}(1-e^2)^{\frac{1}{2}}\,\{(N_2^2-N_1^2)\,\cos 2\omega - 2\,N_1\,N_2\,\sin 2\omega\} \qquad (144)$$

and

$$\frac{d\,\bar{\sigma}_{-3,2}}{dt} = -\frac{15\mu_d\,(1-e_d^2)^{-3/2}}{8\,a_d^3\,n}(1+e^2)\,\{(N_2^2-N_1^2)\,\cos 2\omega - 2\,N_1\,N_2\,\sin 2\omega\}\ . \qquad (145)$$

For $n = -4$ and $k = 1$,

$$\frac{d\,\bar{e}_{-4,1}}{dt} = -\frac{15\mu_d\, e_d(1-e_d^2)^{-5/2}}{64\, a_d^4\, n}\, a(1-e^2)^{\frac{1}{2}}\left(1+\frac{3}{4}e^2\right)$$

$$\times \{\sin\omega_d \sin\omega\,(4M_1-5L_3N_2+15M_3N_1N_3) + \cos\omega_d \sin\omega\,(4L_1+5M_3N_2+15L_3N_1N_3)$$

$$- \sin\omega_d \cos\omega\,(4M_2+5L_3N_1+15M_3N_2N_3) - \cos\omega_d \cos\omega\,(4L_2-5M_3N_1+15L_3N_2N_3)\}\,,$$

$$\ldots\ldots \quad (146)$$

$$\frac{d\,\bar{I}_{-4,1}}{dt} = \frac{15\mu_d\, e_d(1-e_d^2)^{-5/2}}{32\, a_d^4\, n}\, ae(1-e^2)^{-\frac{1}{2}}\left(1+\frac{3}{4}e^2\right)$$

$$\times \Big\{5\sin\omega_d \sin\omega\,(M_1N_2N_3+M_2N_1N_3+M_3N_1N_2) + 5\cos\omega_d \sin\omega\,(L_1N_2N_3+L_2N_1N_3 + L_3N_1N_2)$$

$$+ \sin\omega_d \cos\omega\left(\frac{15}{2}M_3N_2^2 + \frac{45}{2}M_3N_1^2 - 7M_3 - 10L_2N_1\right)$$

$$+ \cos\omega_d \cos\omega\left(\frac{15}{2}L_3N_2^2 + \frac{45}{2}L_3N_1^2 - 7L_3 + 10M_2N_1\right)\Big\}\,, \quad (147)$$

$$\frac{d\,\bar{\Omega}_{-4,1}}{dt} = \frac{15\mu_d\, e_d(1-e_d^2)^{-5/2}}{32\, a_d^4\, n \sin i}\, ae(1-e^2)^{-\frac{1}{2}}\left(1+\frac{3}{4}e^2\right)$$

$$\times \Big\{\sin\omega_d \sin\omega\left(\frac{15}{2}M_3N_1^2 + \frac{45}{2}M_3N_2^2 - 7M_3 + 10L_1N_2\right)$$

$$+ \cos\omega_d \sin\omega\left(\frac{15}{2}L_3N_1^2 + \frac{45}{2}L_3N_2^2 - 7L_3 - 10M_1N_2\right)$$

$$+ 5\sin\omega_d \cos\omega\,(M_1N_2N_3+M_2N_1N_3+M_3N_1N_2)$$

$$+ 5\cos\omega_d \cos\omega\,(L_1N_2N_3+L_2N_1N_3+L_3N_1N_2)\Big\}\,, \quad (148)$$

$$\frac{d\,\bar{\omega}_{-4,1}}{dt} + \cos i\,\frac{d\,\bar{\Omega}_{-4,1}}{dt} = -\frac{15\mu_d\,e_d(1-e_d^2)^{-5/2}}{64\,a_d^4\,n}\,ae^{-1}(1-e^2)^{\frac{1}{2}}\left(1+\frac{9}{4}e^2\right)$$

$$\times\{\sin\omega_d\,\sin\omega\,(4M_2+5L_3N_1+15M_3N_2N_3)$$
$$+\cos\omega_d\,\sin\omega\,(4L_2-5M_3N_1+15L_3N_2N_3)$$
$$+\sin\omega_d\,\cos\omega\,(4M_1-5L_3N_2+15M_3N_1N_3)$$
$$+\cos\omega_d\,\cos\omega\,(4L_1+5M_3N_2+15L_3N_1N_3)\} \qquad (149)$$

and

$$\frac{d\,\bar{\sigma}_{-4,1}}{dt} = \frac{15\mu_d\,e_d(1-e_d^2)^{-5/2}}{64\,a_d^4\,n}\,ae^{-1}\left(1+\frac{29}{4}e^2+\frac{9}{4}e^4\right)$$

$$\times\{\sin\omega_d\,\sin\omega\,(4M_2+5L_3N_1+15M_3N_2N_3)+\cos\omega_d\,\sin\omega\,(4L_2-5M_3N_1+15L_3N_2N_3)$$
$$+\sin\omega_d\,\cos\omega\,(4M_1-5L_3N_2+15M_3N_1N_3)+\cos\omega_d\,\cos\omega\,(4L_1+5M_3N_2+15L_3N_1N_3)\}.$$

$$\qquad \ldots. \quad (150)$$

For $n = -4$ and $k = 3$,

$$\frac{d\,\bar{e}_{-4,3}}{dt} = \frac{1575\mu_d\,e_d(1-e_d^2)^{-5/2}}{256\,a_d^4\,n}\,ae^2(1-e^2)^{\frac{1}{2}}$$

$$\times\{\sin\omega_d\,\sin 3\omega\,(3L_3N_2-M_3N_1N_3-4M_1N_2^2)-\cos\omega_d\,\sin 3\omega\,(3M_3N_2+L_3N_1N_3+4L_1N_2^2)$$
$$-\sin\omega_d\,\cos 3\omega\,(3L_3N_1+M_3N_2N_3+4M_2N_1^2)+\cos\omega_d\,\cos 3\omega\,(3M_3N_1-L_3N_2N_3$$
$$-4L_2N_1^2)\},$$

$$\ldots.(151)$$

$$\frac{d\,\bar{i}_{-4,3}}{dt} = \frac{525\mu_d\, e_d(1-e_d^2)^{-5/2}}{256\, a_d^4\, n}\, ae^3(1-e^2)^{-\frac{1}{2}}$$

$$\times \{2\sin\omega_d \sin 3\omega\, (L_1N_1-L_2N_2+3M_3N_1N_2) - 2\cos\omega_d \sin 3\omega\, (M_1N_1-M_2N_2-3L_3N_1N_2)$$
$$+ \sin\omega_d \cos 3\omega\, (2M_3+3M_3N_1^2-3M_3N_2^2-4L_2N_1)$$
$$- \cos\omega_d \cos 3\omega\, (2L_3-3L_3N_1^2+3L_3N_2^2-4M_1N_2)\} \quad , \qquad (152)$$

$$\frac{d\,\bar{\Omega}_{-4,3}}{dt} = \frac{525\mu_d\, e_d(1-e_d^2)^{-5/2}}{256\, a_d^4\, n \sin i}\, ae^3(1-e^2)^{-\frac{1}{2}}$$

$$\times \{\sin\omega_d \sin 3\omega\, (2M_3+3M_3N_1^2-3M_3N_2^2-4L_2N_1)$$
$$- \cos\omega_d \sin 3\omega\, (2L_3-3L_3N_1^2+3L_3N_2^2-4M_1N_2)$$
$$- 2\sin\omega_d \cos 3\omega\, (L_1N_1-L_2N_2+3M_3N_1N_2)$$
$$+2\cos\omega_d \cos 3\omega\, (M_1N_1-M_2N_2-3L_3N_1N_2)\} \quad , \qquad (153)$$

$$\frac{d\,\bar{\omega}_{-4,3}}{dt} + \cos i\, \frac{d\,\bar{\Omega}_{-4,3}}{dt} = \frac{1575\mu_d\, e_d(1-e_d^2)^{-5/2}}{256\, a_d^4\, n}\, ae(1-e^2)^{\frac{1}{2}}$$

$$\times \{\sin\omega_d \sin 3\omega\, (3L_3N_1+M_3N_2N_3+4M_2N_1^2)$$
$$- \cos\omega_d \sin 3\omega\, (3M_3N_1-L_3N_2N_3-4L_2N_1^2)$$
$$+ \sin\omega_d \cos 3\omega\, (3L_3N_2-M_3N_1N_3-4M_1N_2^2)$$
$$- \cos\omega_d \cos 3\omega\, (3M_3N_2+L_3N_1N_3+4L_1N_2^2)\} \qquad (154)$$

and

$$\frac{d\,\sigma_{-4,3}}{dt} = -\frac{1575\mu_d\, e_d(1-e_d^2)^{-5/2}}{256\, a_d^4\, n}\, ae(1+e^2)$$

$$\times \{\sin\omega_d \sin 3\omega\, (3L_3N_1+M_3N_2N_3+4M_2N_1^2)$$
$$- \cos\omega_d \sin 3\omega\, (3M_3N_1-L_3N_2N_3-4L_2N_1^2)$$
$$+ \sin\omega_d \cos 3\omega\, (3L_3N_2-M_3N_1N_3-4M_1N_2^2)$$
$$- \cos\omega_d \cos 3\omega\, (3M_3N_2+L_3N_1N_3+4L_1N_2^2)\} \quad . \qquad (155)$$

For $n = -5$ and $k = 0$,

$$\frac{d\,\bar{e}_{-5,0}}{dt} = 0 \quad , \tag{156}$$

$$\frac{d\,\bar{i}_{-5,0}}{dt} = \frac{135\mu_d\,(1-e_d^2)^{-7/2}}{128\,a_d^5\,n}\,a^2(1-e^2)^{-\frac{1}{2}}\left(1+5e^2+\frac{15}{8}e^4\right)$$
$$\times\Bigg[\left(1+\frac{3}{2}e_d^2\right)N_1N_3\left(1-\frac{7}{3}N_3^2\right)+e_d^2\left\{L_1L_3-M_1M_3+\frac{7}{3}L_3M_3N_2+\frac{7}{3}N_1N_3\,(L_3^2-M_3^2)\right\}\cos 2\,\omega_d$$
$$+\,e_d^2\left\{L_1M_3+L_3M_1+\frac{14}{3}L_3M_3N_1N_3+\frac{7}{6}N_2(M_3^2-L_3^2)\right\}\sin 2\omega_d\Bigg] \quad , \tag{157}$$

$$\frac{d\,\bar{\Omega}_{-5,0}}{dt} = \frac{135\mu_d\,(1-e_d^2)^{-7/2}}{128\,a_d^5\,n\,\sin i}\,a^2(1-e^2)^{-\frac{1}{2}}\left(1+5e^2+\frac{15}{8}e^4\right)$$
$$\times\Bigg[\left(1+\frac{3}{2}e_d^2\right)N_2N_3\left(1-\frac{7}{3}N_3^2\right)+e_d^2\left\{L_2L_3-M_2M_3-\frac{7}{3}L_3M_3N_1+\frac{7}{3}N_2N_3\,(L_3^2-M_3^2)\right\}\cos 2\,\omega_d$$
$$+\,e_d^2\left\{L_2M_3+L_3M_2+\frac{14}{3}L_3M_3N_2N_3+\frac{7}{6}N_1(L_3^2-M_3^2)\right\}\sin 2\omega_d\Bigg] \quad , \tag{158}$$

$$\frac{d\,\bar{\omega}_{-5,0}}{dt}+\cos i\,\frac{d\,\bar{\Omega}_{-5,0}}{dt} = \frac{135\mu_d\,(1-e_d^2)^{-7/2}}{256\,a_d^5\,n}\,a^2(1-e^2)^{\frac{1}{2}}\left(1+\frac{3}{4}e^2\right)$$
$$\times\Bigg[\left(1+\frac{3}{2}e_d^2\right)\left(1-10N_3^2+\frac{35}{3}N_3^4\right)+\frac{5}{3}e_d^2\,(1-7N_3^2)$$
$$\times\left\{(L_3^2-M_3^2)\,\cos 2\,\omega_d+2L_3M_3\,\sin 2\,\omega_d\right\}\Bigg] \tag{159}$$

and

$$\frac{d\,\bar{\sigma}_{-5,0}}{dt} = -\,\frac{243\mu_d\,(1-e_d^2)^{-7/2}}{256\,a_d^5\,n}\,a^2\left(1+\frac{25}{12}e^2+\frac{5}{12}e^4\right)$$
$$\times\Bigg[\left(1+\frac{3}{2}e_d^2\right)\left(1-10N_3^2+\frac{35}{3}N_3^4\right)+\frac{5}{3}e_d^2\,(1-7N_3^2)$$
$$\times\left\{(L_3^2-M_3^2)\,\cos 2\,\omega_d+2L_3M_3\,\sin 2\,\omega_d\right\}\Bigg] \quad . \tag{160}$$

These expressions have been compared with those given by Smith[25] as far as comparison is possible - Ref.25 neglects cubic and higher powers of a/a_d, e and e_d. Most of the terms compared agree but there are some discrepancies for $d\bar{\omega}/dt$, as there were for $\Delta\omega$ in section 8.3, and one discrepancy for each of $d\bar{e}/dt$ and $d\bar{I}/dt$. In a private communication Smith has agreed that the errors are in Ref.25.

The details of the algebra required to reach the above expressions have been omitted. The analysis is fairly straightforward, but one point of difficulty is worth mentioning. In applying equations (131), (132), (134) and (135) to eliminate α and θ_d, negative powers of $\sin^2 i''$ are temporarily introduced. They must be cancelled by factorisation of the polynomial expression in L_3, M_3, N_1, N_2 and N_3 which also appears. That this factorisation is possible is not always evident and an example is now given of the factorisation for one of the polynomial expressions. The method employed is to eliminate N_1, N_2 and N_3 first, using relations of the type $N_1^2 = 1-L_1^2-M_1^2$, $N_1N_3 = -(L_1L_3+M_1M_3)$ and $N_2 = L_3M_1-L_1M_3$, then to regroup terms, to factorise and finally to simplify by re-introducing N_1, N_2 and N_3.

As our example we extract the factor $\sin^2 i''$ $(= L_3^2+M_3^2)$ from the expression $L_3N_2^3+M_3N_1^3N_3$ which occurs, multiplied by $\sin\omega_d \sin 3\omega/\sin^2 i''$, in the evaluation of $d\bar{e}_{-4,3}/dt$. We have

$$\begin{aligned}
L_3N_2^3+M_3N_1^3N_3 &= L_3(L_3M_1-L_1M_3)(1-L_2^2-M_2^2) - M_3(L_1L_3+M_1M_3)(1-L_1^2-M_1^2) \\
&= L_1L_3M_3(L_1^2+M_1^2+L_2^2+M_2^2-2) + M_1\{L_3^2(1-L_2^2-M_2^2) - M_3^2(1-L_1^2-M_1^2)\} \\
&= -L_1L_3M_3(L_3^2+M_3^2) + M_1\{L_3^2(1-L_2^2-M_2^2) - M_3^2(L_3^2+M_3^2) - M_3^2(1-L_1^2-M_1^2-L_3^2-M_3^2)\} \\
&= -L_1L_3M_3(L_3^2+M_3^2) - M_1M_3^2(L_3^2+M_3^2) + M_1(1-L_2^2-M_2^2)(L_3^2+M_3^2) \\
&= (L_3^2+M_3^2)(M_1N_2^2-L_1L_3M_3-M_1M_3^2) \\
&= (L_3^2+M_3^2)(M_1N_2^2+M_3N_1N_3) \quad .
\end{aligned}$$

9 APPLICATION TO SOLAR RADIATION PRESSURE (NO SHADOW)

It is interesting to note that, of the results for the complete range of values of n ($-\infty$ to $+\infty$) covered by the general theory of sections 4 to 6, interpretations have been found for every case but one; for n = 2 to ∞ the results apply to the gravitational field outside the earth, for n = 1,0 and -1

the results are trivial and have been interpreted in section 6.1, and for $n = -3$ to $-\infty$ the results apply to luni-solar (gravitational) perturbations. This leaves the results for $n = -2$ unused and it so happens that it is just these results which are relevant to the study of the perturbations due to solar radiation pressure.

It is necessary to assume that the satellite remains in sunlight, since in shadow the radiation pressure disappears abruptly. Equivalently we may think of the earth as transparent, so that it is possible to have a conservative field everywhere. Then, neglecting any asymmetry in the shape or surface finish of the satellite, and taking the sun to be at an infinite distance, the radiation pressure is a constant force - directed always away from the sun. The direction of the sun is an axis of symmetry and the potential function is given by

$$U = Fr \cos \varphi \quad , \tag{161}$$

where φ is the angle, subtended at the centre of the earth, between the satellite and the sun; F is the (constant) force per unit mass towards the sun - using the notation of Cook[2] - and is always negative.

Thus

$$U = U_{-2}$$

on taking $J_{-2} = - F R^2/\mu$; here J_{-2} is positive and the interpretation of the distance R is unnecessary since it will be cancelled out in obtaining the perturbations $\Delta\zeta$.

The treatment of section 8 is now immediately applicable (with $n = -2$) and there are again two cases, according to whether or not we average over a year. If the sun is assumed fixed we use the A,B,C of section 8.3. Instead of working from the general formulae we take advantage of the formulae for ζ_{-2} in section 6.4. It is necessary to bear in mind that these equations must be taken as referring to the dashed elements of section 8.3. To obtain normal elements we use equations (74) and (79) to (82). Since

$$\Delta\zeta = 2\pi J_{-2} \frac{a^2}{R^2} \zeta_{-2} = - \frac{2\pi F a^2}{\mu} \zeta_{-2} \quad ,$$

we get, very easily, the following results:

$$\Delta a = 0 \quad , \tag{162}$$

$$\Delta e = \frac{3\pi F}{\mu} a^2(1-e^2)^{\frac{1}{2}} (B\cos\omega - A\sin\omega) \quad , \tag{163}$$

$$\Delta i = -\frac{3\pi F}{\mu} a^2 e(1-e^2)^{\frac{1}{2}} C\cos\omega \quad , \tag{164}$$

$$\sin i \, \Delta\Omega = -\frac{3\pi F}{\mu} a^2 e(1-e^2)^{\frac{1}{2}} C\sin\omega \quad , \tag{165}$$

$$\Delta\omega + \cos i \, \Delta\Omega = -\frac{3\pi F}{\mu} a^2 e^{-1}(1-e^2)^{\frac{1}{2}} (A\cos\omega + B\sin\omega) \tag{166}$$

and

$$\Delta\sigma = \frac{3\pi F}{\mu} a^2(e^{-1}+e) \ (A\cos\omega + B\sin\omega) \quad . \tag{167}$$

The results of equations (163) to (166) agree with those given, in a different form, by Allan - equations (14) of Ref. 24.

Passing to the consideration of the mean effects of solar radiation over one year, it is advisable to interpret R as the (variable) distance of the sun from the earth. If we then assume that $F R^2 = \text{const} = F_o a_d^2$, where a_d is the semi-major axis of the sun's orbit, we may pursue the analysis of section 8.4. Double-dashed elements are based on the ecliptic as reference plane.

With $n = -2$, k takes the value 1 only. On replacing μ_d by $F_o a_d^2$ equation (88) gives

$$U_{-2}(av) = -\frac{3}{2} F_o \ ae \left(\frac{a_d}{R}\right)^2 (\sin\omega'' \cos i'' \sin(\theta_d+v_d) + \cos\omega'' \cos(\theta_d+v_d)) \quad . \qquad \text{....} \ (168)$$

Hence

$$U_{-2}(av, av) = -\frac{3}{2} F_o \ ae(1-e_d^2)^{-\frac{1}{2}} \frac{1}{2\pi} \int_0^{2\pi} \{\sin\omega'' \cos i'' \sin(\theta_d+v_d) + \cos\omega'' \cos(\theta_d+v_d)\} \, d v_d$$

$$= 0 \quad .$$

Thus

$$\frac{d \xi_{-2}}{dt} = 0 \quad . \tag{169}$$

for every element.

10 SECULAR AND LONG-PERIOD PERTURBATIONS

10.1 Introductory remarks

The bulk of this paper has been devoted to the first-order perturbations of a satellite orbit over a period of one revolution of the satellite. The perturbations have been 'first-order' in that each term of the perturbing potential U, given by equation (11), is treated quite separately. The integration of Lagrange's equations (2) to (7) has been carried out with the elements on the right hand sides held constant. Perturbations over many revolutions are assumed to be given by multiples of the perturbations over one revolution.

This treatment would be adequate for dealing with the potential field of the earth if it were the case that each J_n $(n > 2)$ were of the order 10^{-10} say. The actual position, however, is that J_2 is $O(10^{-3})$ and the other J_n, with $n > 2$, are $O(10^{-6})$. Thus it is in some ways more logical to regard J_2 terms, only, as first-order and the terms for later J_n, together with J_2^2 terms, as second-order. Up to the present we have preferred, for the sake of generality, not to do this, but now it is desirable to consider matters in the light of the actual geopotential field.

The key effects of the dominant J_2 term are on Ω and ω, these elements being given secular, i.e. non-periodic, changes. If the rates of secular motion are denoted by $\dot{\Omega}$ and $\dot{\omega}$ respectively - the dot notation is used nowhere else in this Report - then

$$\dot{\Omega} = J_2\, n \left(\frac{R}{p}\right)^2 \Omega_{20} \qquad \text{and} \qquad \dot{\omega} = J_2\, n \left(\frac{R}{p}\right)^2 \omega_{20} ,$$

where n is the mean motion and Ω_{20}, ω_{20} are given by section 6.3; thus

$$\dot{\Omega} = -\frac{3}{2} J_2\, n \left(\frac{R}{p}\right)^2 \cos i \tag{170}$$

and

$$\dot{\omega} = 3 J_2\, n \left(\frac{R}{p}\right)^2 \left(1 - \frac{5}{4} f\right) \tag{171}$$

where $f = \sin^2 i$. The element σ also has a secular component, $\dot{\sigma}$, but this has not the importance of $\dot{\Omega}$ and $\dot{\omega}$.

In section 10.2 we consider long-period effects arising from the interaction between $\dot{\omega}$ and the (previously-called) first-order perturbations associated with J_3, J_4 etc; J_2^2 perturbations are introduced and their interaction with $\dot{\omega}$ considered also. In section 10.3 we consider, similarly, the interaction of $\dot{\Omega}$ and $\dot{\omega}$ with luni-solar perturbations.

These introductory remarks are concluded by a brief mention of short-period perturbations. Until now these have not arisen because Lagrange's planetary equations have always been integrated over a complete revolution of the satellite. Short-period terms are those which arise during a revolution but vanish at its end. The only short-period terms which are significant are those associated with J_2. A good way to derive them is to integrate the Lagrangian equations using true anomaly v as argument instead of mean anomaly M.

Merson[1] has applied this method and his results (for the standard osculating elements) are given for integration from the ascending node of the satellite orbit as initial point. Sterne[29] gives results which apply for integration from any point of the orbit, the node being one obvious choice and perigee another. We follow Sterne and list here, for reference, the complete J_2 perturbations - integrated to a general point of the orbit - for each of our six standard elements, using v as argument. The elements on the right hand sides of equations (172) to (177) should be taken to refer to conditions at the chosen initial point of the orbit (when $t = 0$) and the functions $F(e,i,\omega)$ must be chosen to make the expressions vanish at this point; e.g. for integration from the node, $F_{\Omega}(e,i,\omega) = 6\omega + 8\, e \sin\omega$. The expressions agree with those given by Merson and Sterne. To obtain purely short-period terms, it is only necessary - for Ω, ω and σ - to replace the initial secular v by v-M.

$$\begin{aligned}\delta a = \frac{J_2 R^2 a^2}{16\, p^3}\big[&12\,(2+3e^2) f \cos 2(\omega+v) + 6e\,(4+e^2)(2-3f)\cos v \\ &+ 9e\,(4+e^2) f\{\cos(2\omega+v)+\cos(2\omega+3v)\} + 12e^2\,(2-3f)\cos 2v \\ &+ 18e^2 f \cos 2(\omega+2v) + 3e^3 f\{\cos(2\omega-v)+\cos(2\omega+5v)\} \\ &+ 2e^3\,(2-3f)\cos 3v - F_a(e,i,\omega)\big] \quad , \qquad (172)\end{aligned}$$

$$\begin{aligned}\delta e = \frac{J_2 R^2}{32\, p^2}\big[&6\,(4+e^2)(2-3f)\cos v + 3(4+11e^2) f \cos(2\omega+v) \\ &+ (28+17e^2) f \cos(2\omega+3v) \\ &+ 12e(2-3f)\cos 2v + 6ef\{10\cos 2(\omega+v) + 3\cos 2(\omega+2v)\} \\ &+ 3e^2 f\{\cos(2\omega-v)+\cos(2\omega+5v)\} + 2e^2(2-3f)\cos 3v - F_e(e,i,\omega)\big] ,\\ & \qquad .... \quad (173)\end{aligned}$$

$$\delta i = \frac{J_2 R^2 \sin 2i}{8 p^2}[3\cos 2(\omega+v) + 3e\cos(2\omega+v) + e\cos(2\omega+3v) - F_i(e,i,\omega)] , \tag{174}$$

$$\delta\Omega = \frac{J_2 R^2 \cos i}{4 p^2}[-6v + 3\sin 2(\omega+v) - 6e\sin v + 3e\sin(2\omega+v) + e\sin(2\omega+3v) - F_\Omega(e,i,\omega)] , \tag{175}$$

$$\begin{aligned}\delta\omega = \frac{J_2 R^2}{32 p^2}[&24(4-5f)v + 6e^{-1}\{4(2-3f)+e^2(14-17f)\}\sin v \\ &- 3e^{-1}\{4f + e^2(8-15f)\}\sin(2\omega+v) + e^{-1}\{28f-e^2(8-19f)\} \times \sin(2\omega+3v) \\ &+ 12(2-3f)\sin 2v - 12(2-5f)\sin 2(\omega+v) + 18f\sin 2(\omega+2v) \\ &- 3ef\sin(2\omega-v) + 2e(2-3f)\sin 3v + 3ef\sin(2\omega+5v) \\ &- F_\omega(e,i,\omega)]\end{aligned} \tag{176}$$

and

$$\begin{aligned}\delta\sigma = \frac{J_2 R^2(1-e^2)^{\frac{1}{2}}}{32 p^2}[&24(2-3f)v - 6e^{-1}(4-5e^2)(2-3f)\sin v \\ &+ 3e^{-1}(4+17e^2)f\sin(2\omega+v) - e^{-1}(28-13e^2)f\sin(2\omega+3v) - 12(2-3f)\sin 2v \\ &+ 36f\sin 2(\omega+v) - 18f\sin 2(\omega+2v) + 3ef\sin(2\omega-v) \\ &- 2e(2-3f)\sin 3v - 3ef\sin(2\omega+5v) - F_\sigma(e,i,\omega)] .\end{aligned}$$

.... (177)

It is to be noted that Sterne does not give, as his sixth expression, a formula like equation (177). He gives, instead, a formula which is more convenient for the derivation of M, the quantity actually required. From equation (8),

$$\delta M = \delta\sigma + \int_0^t nt \quad ;$$

using the constant n' introduced in section 7 and equations (63) and (65), with the indicial suffix n = 2, we then have

$$\delta M = n't + \delta\sigma - \tfrac{1}{2}\delta p \quad . \tag{178}$$

With the expression for $\delta\rho$ available (it is $-\delta\varphi$ of Merson[1]) we get

$$\delta M = n't - \frac{J_2 R^2(1-e^2)^{\frac{1}{2}}}{32\,p^2}[6e^{-1}(4-e^2)(2-3f)\sin v - 3e^{-1}(4+5e^2)f\sin(2\omega+v) + e^{-1}(28-e^2)f\sin(2\omega+3v) + 12(2-3f)\sin 2v + 18f\sin 2(\omega+2v) - 3ef\sin(2\omega-v) + 2e(2-3f)\sin 3v + 3ef\sin(2\omega+5v) - F_M(e,i,\omega)] \,. \quad \ldots\ldots (179)$$

For the pure short-period behaviour the $n't$ term must, of course, be dropped.

The $\frac{1}{2}\,\delta\rho$ term in equation (178) does not arise if Merson's smoothed elements are used instead of osculating elements. The short-period expressions of equations (172) to (177) then become very much simpler; in particular, terms in e^{-1} disappear, though we have already - in section 8.3 - made the point that such terms are not of undue significance.

10.2 Secular and long-period perturbations associated with J_3, J_4 etc and with J_2^2

There is a conceptually straightforward method, associated with the names of Poisson and Poicaré, which it is possible to apply in order to obtain perturbations associated with successively higher powers of the J coefficients. The method has been applied by Merson[1] to obtain the J_2^2 perturbations, but already at the expense of much laborious calculation. For higher powers, e.g. $J_2\,J_3$, it becomes quite unmanageable, but it is nevertheless worth mentioning briefly.

The principle of the method is an iterative one based on the assumption that the complete perturbation of each element, ζ, may be expressed - in terms of the elements a_o, e_o, i_o and ω_o at epoch ($t = 0$) and of the true anomaly v - by a Taylor-type series of the form

$$\delta\zeta = \sum_{S_j} J_2^{S_2} J_3^{S_3} J_4^{S_4} \ldots\ldots f_{S_2,S_3,\ldots\ldots}(a_o, e_o, i_o, \omega_o, v) \,, \quad (180)$$

where v is not confined to values within one revolution of $t = 0$ (we ignore convergence questions, of which there is a short discussion at the end of section 4.6 of Ref.29). For $S_2 = 1$ and $S_3 = S_4 = \ldots\ldots = 0$, the functions f are precisely those implied by equations (172) to (177). For $S_3 = 1$ and

$S_2 = S_4 = \dots = 0$, the full first-order J_3 perturbation could be obtained; and so on, Lagrange's equations for these first-order perturbations being integrated with the elements on the right hand sides held fixed at a_0, e_0 etc. Second-order perturbations (with $\Sigma S_j = 2$) could then be obtained by substituting, on the right hand sides of the Lagrangian equations, functions of the elements with first-order perturbations included. Proceeding in this way perturbations of any order are in principle determinate.

Merson[1] has used this method to obtain J_2^2 terms for osculating elements. Although he does not derive short-period terms (for which a paper of Kozai[30] may be consulted) the method is already extremely laborious at the J_2^2 level. The Merson J_2^2 expressions are given for the variation from node to node; from perigee to perigee the variations are different (unlike the situation for J_3, J_4 etc variations) due to the J_2 variation in ω. The perigee-to-perigee expressions are used later in the section and are listed below; $\Delta\zeta_\omega$ is the second-order change associated with the first-order change given for J_2 by equation (36) with elements in equations (38) to (45) taken at perigee.

$$\Delta a_\omega = -\frac{9\pi}{2} J_2^2 R^4 a^2 p^{-5} (1+e)^3 f(4-5f) \sin 2\omega \quad , \tag{181}$$

$$\Delta e_\omega = -\frac{3\pi}{16} J_2^2 \left(\frac{R}{p}\right)^4 (1+e) f \{4(4-5f)(5+7e) + e(1-e)(14-15f)\} \sin 2\omega \quad , \tag{182}$$

$$\Delta i_\omega = \frac{3\pi}{32} J_2^2 \left(\frac{R}{p}\right)^4 \sin 2i \{(14-15f) e^2 - 4(4-5f)(3+4e)\} \sin 2\omega \quad , \tag{183}$$

$$\Delta\Omega_\omega = -\frac{3\pi}{16} J_2^2 \left(\frac{R}{p}\right)^4 \cos i \, [4(9-10f) + e^2(4+5f) - \{8(3+4e)(2-5f) - 2e^2(7-15f)\} \cos 2\omega] \quad , \quad \dots \tag{184}$$

$$\Delta\omega_\omega = \frac{3\pi}{64} J_2^2 \left(\frac{R}{p}\right)^4 [2(192-412f+215f^2) + e^2 (56-36f-45f^2) + \{32e^{-1} f(4-5f) - 4(48-322f+315f^2) - 64e(4-26f+25f^2) + 2e^2(28-158f+135f^2)\} \cos 2\omega] \tag{185}$$

and

$$\Delta\sigma_\omega = \frac{3\pi}{32} J_2^2 \left(\frac{R}{p}\right)^4 (1-e^2)^{-\frac{1}{2}} \Big[2(48-116f+67f^2) + 12e(2-3f)^2 - 2e^2(16-52f+59f^2) + 4e^3(2-3f)^2 - e^4\left(35-41f-\frac{79}{8}f^2\right)$$
$$-f\{16e^{-1}(4-5f)-2(170-213f)-4e(146-187f) + 8e^2(49-57f)+4e^3(106-131f)-e^4(106-111f)\}\cos 2\omega$$
$$-\frac{9}{16}e^4 f^2 \cos 4\omega \Big] \quad . \qquad (186)$$

Some general, but lengthy, expressions for J_2^2 perturbations have been given by Claus and Lubowe[31]. These expressions refer to a complete revolution of the satellite from $v+\omega = \theta_o$ to $\theta_o+2\pi$, using the notation of Ref. 31. For $\theta_o = 0$, Merson's expressions are confirmed. For $\theta_o = \omega_o$, however, v changes from 0 to $2\pi-\Delta\omega$ and so we do not get our equations (181) to (186). It is interesting to observe that for this case $(\theta_o = \omega_o)$ Ref. 31 leads to a J_2^2 perturbation of semi-major axis equal to zero.

The general method outlined above is impractical then. The following point about it should be noted, however; its significance will appear later. Just as (secular) terms in v appear in the perturbations associated with J_2, J_3, ... and J_2^2, there similarly appear - as dominant terms over a long interval - terms in v^2 associated with $J_2 J_3$, $J_2 J_4$, ... and J_2^3; for still higher powers of the J coefficients there appear higher powers of v.

A suitable treatment, which takes account of the main higher-order terms, is in fact rather obvious and will now be described. Leaving J_2^2 terms until the end, we consider the perturbations due to J_3, J_4, etc; we work generally using J_ℓ, where ℓ is appropriate as the general suffix since only positive values are required and we want n to denote the mean motion. Since there is no long-period or secular variation of semi-major axis we start by considering eccentricity, e.

We eliminate short-period terms by using 'barred' elements and write, from equations (36) and (37),

$$\frac{d\bar{e}}{dt} = \frac{n}{2\pi}\Delta e = n\sum_{\ell=2}^{\infty}\sum_{k=0}^{\ell} J_\ell \left(\frac{R}{p}\right)^\ell e_{\ell k} \quad . \qquad (187)$$

Hence, abbreviating equation (39) to

$$e_{\ell k} = G_{\ell k}(e,i)\sin k(\omega-\tfrac{1}{2}\pi) \quad ,$$

where the function $G_{\ell k}(e,i)$ is thereby defined,

$$\frac{d\bar{e}}{dt} = n\sum_{\ell,k} J_\ell \left(\frac{R}{p}\right)^\ell G_{\ell k}(e,i)\sin k(\omega-\tfrac{1}{2}\pi) .$$

Now our interest is in the interaction of J_ℓ ($\ell > 2$) with J_2. Since a, e and i are constant to order J_2, the variation in $\bar{e}$ is given by

$$\delta\bar{e} = n\sum_{\ell,k} J_\ell \left(\frac{R}{p}\right)^\ell G_{\ell k}(e,i)\int_0^t \sin k(\omega-\tfrac{1}{2}\pi)dt .$$

Taking the J_2 variation of ω to be given by equation (171), we may write (since only terms with $k > 0$ occur)

$$\delta\bar{e} = -\sum_{\ell,k} \frac{nJ_\ell}{k\dot{\omega}}\left(\frac{R}{p}\right)^\ell G_{\ell k}(e,i)\,\{\cos k(\omega-\tfrac{1}{2}\pi) - \cos k(\omega_0-\tfrac{1}{2}\pi)\} . \quad (188)$$

Here t does not appear so that $\delta\bar{e}$ is not secular but long-periodic (by implied definition of 'long-periodic').

Two points may be observed about equation (188). They apply also to the perturbations for the other elements.

First, if $\cos k(\omega-\frac{1}{2}\pi)$ is expanded by Taylor's theorem (valid for ω of any magnitude) with $\omega-\omega_0 = \frac{\dot{\omega}}{n}(v-v_0)$, short-period terms being neglected, then

$$-\frac{n}{k\dot{\omega}}\{\cos k(\omega-\tfrac{1}{2}\pi) - \cos k(\omega_0-\tfrac{1}{2}\pi)\} = (v-v_0)\sin k(\omega_0-\tfrac{1}{2}\pi) + \tfrac{1}{2}\frac{k\dot{\omega}}{n}(v-v_0)^2\cos k(\omega_0-\tfrac{1}{2}\pi) + \ldots\ldots .$$

With the first term here we have recovered the appropriate contribution to $\delta\zeta$ in equation (180) with $S_\ell = 1$ and the other $S_j = 0$. With the second term we have precisely the dominant (quadratic) term of those associated with J_2J_ℓ, of which the significance was forecast. Similarly, successive terms are associated with dominant terms in v^3, v^4 etc associated with $J_2^2 J_\ell$, $J_2^3 J_\ell$ etc.

The second point to be observed about equation (188) is that the lower limit ω_0, corresponding to $t = 0$, appears in it. The term in ω_0 is often omitted for simplicity, for example in the work of Kozai[12]. The variation in e is then given from its value at some time for which ω is such that $\cos k(\omega-\frac{1}{2}\pi) = 0$. The present author feels that one may then be only too easily confused by what happens when $\dot{\omega}$ is small. This occurs when $\sin^2 i$ is close to

4/5; i.e. when i is about 63°.4 or 116°.6, the so-called critical inclinations. If the ω_o term is dropped from equation (188), then when $\dot{\omega}$ is small the value of $\delta\bar{e}_{\ell k}$ is in general going to be large (relative to J_ℓ) and there appears to be a singularity. The trouble is, of course, that when $\dot{\omega}$ is small the perigee is moving so slowly that the time at which $\cos k\,(\omega-\frac{1}{2}\pi) = 0$, if it exists at all, may be a long way from the time at which the perturbation is of interest. Many papers have been devoted to the subject of the critical inclinations, e.g. Refs.32, 33 and 34, and some rather advanced mathematics is involved. It is perhaps sometimes overlooked, however, that the situation is in some ways simpler than for non-critical inclinations. For if $\dot{\omega}$ is zero, ω is no longer a different type of element from e and i in its J_2 behaviour. The integration of equation (187) then leads simply to

$$\delta\bar{e} = n \sum_{\ell,k} J_\ell \left(\frac{R}{p}\right)^{\ell} e_{\ell k}\, t \quad ,$$

to which equation (188) reduces as $\dot{\omega} \to 0$.

The long-period perturbation to the element i, corresponding to equation (188), is given by

$$\delta\bar{i} = \sum_{\ell,k} \frac{nJ_\ell}{k\dot{\omega}} \left(\frac{R}{p}\right)^{\ell} e(1-e^2)^{-1} \cot i\; G_{\ell k}(e,i) \left\{\cos k\,(\omega-\tfrac{1}{2}\pi) - \cos k\,(\omega_o-\tfrac{1}{2}\pi)\right\}. \quad (189)$$

For the elements Ω, ω and σ there is a complication arising from their main J_2 secular perturbations. For Ω, we start by writing (cf. equation (41))

$$\frac{d\bar{\Omega}}{dt} = n \sum_{\ell,k} J_\ell \left(\frac{R}{p}\right)^{\ell} H_{\ell k}(e,i) \cos k\,(\omega-\tfrac{1}{2}\pi) \quad ,$$

where $H_{\ell k}(e,i) \cos k\,(\omega-\frac{1}{2}\pi) = \Omega_{\ell k}$.

For each $\ell > 2$ and $k = 0$, we get a secular term contributing

$$n\, J_\ell \left(\frac{R}{p}\right)^{\ell} H_{\ell o}(e,i)\, t \qquad (190)$$

to $\delta\bar{\Omega}$. For $k > 0$, we get the long-period contribution, corresponding to equation (188), given by

$$\frac{nJ_\ell}{k\dot{\omega}} \left(\frac{R}{p}\right)^{\ell} H_{\ell k}(e,i) \left\{\sin k\,(\omega-\tfrac{1}{2}\pi) - \sin k\,(\omega_o-\tfrac{1}{2}\pi)\right\} \qquad (191)$$

but it will now be seen that this term may be combined with part of the main secular ($\dot{\Omega}$) term for $\ell = 2$ and $k = 0$. The latter is given by equation (170), which it is necessary to write more precisely, with n' in place of n. We have

$$n' J_2 \left(\frac{R}{p}\right)^2 H_{20}(e,i) = \dot{\Omega} = -\frac{3}{2} n' J_2 \left(\frac{R}{a}\right)^2 (1-e^2)^{-2} \cos i \quad ,$$

so that

$$\frac{\partial\dot{\Omega}}{\partial e} = -6\, n' J_2 \left(\frac{R}{p}\right)^2 e(1-e^2)^{-1} \cos i$$

and

$$\frac{\partial\dot{\Omega}}{\partial i} = \frac{3}{2} n' J_2 \left(\frac{R}{p}\right)^2 \sin i \quad .$$

Hence, using equation (40) and writing n' only where necessary,

$$\dot{\Omega} = -\frac{3}{2} n' J_2 \left(\frac{R}{p_o}\right)^2 \cos i_o - \frac{15}{2} n\, J_2 \left(\frac{R}{p}\right)^2 e(1-e^2)^{-1} \cos i\; \delta\bar{e}$$

and so

$$\int_0^t \dot{\Omega}\, dt = -\frac{3}{2} n' J_2 \left(\frac{R}{p_o}\right)^2 \cos i_o\, t - \frac{15}{2} n\, J_2 \left(\frac{R}{p}\right)^2 e(1-e^2)^{-1} \cos i \int_0^t \delta\bar{e}\, dt \quad .$$

But, from equation (188),

$$\int_0^t \delta\bar{e}\, dt = -\sum_{\ell,k} \frac{n J_\ell}{k\dot{\omega}} \left(\frac{R}{p}\right)^\ell G_{\ell k}(e,i) \left[\frac{1}{k\dot{\omega}} \left\{\sin k\, (\omega - \tfrac{1}{2}\pi) - \sin k\, (\omega_o - \tfrac{1}{2}\pi)\right\} - t \cos k\, (\omega_o - \tfrac{1}{2}\pi)\right] \quad .$$

Hence the combination of the J_2 secular term and the terms of type given by equation (191) - those with $k > 0$ - may be represented by a secular term

$$-\frac{3}{2} n' J_2 \left(\frac{R}{p_o}\right)^2 \cos i_o \left\{1 + 5e(1-e^2)^{-1} \sum_{\ell,k} \frac{n J_\ell}{k\dot{\omega}} \left(\frac{R}{p}\right)^\ell G_{\ell k}(e,i) \cos k\, (\omega_o - \tfrac{1}{2}\pi)\right\} t \qquad (192)$$

together with long-period terms

$$\sum_{\ell,k} \frac{nJ_\ell}{k\dot{\omega}} \left(\frac{R}{p}\right)^\ell \left[H_{\ell k}(e,i) + \frac{5}{2}\,\frac{e(1-e^2)^{-1}\cos i}{k\left(1-\frac{5}{4}f\right)}\,G_{\ell k}(e,i)\right] \{\sin k\,(\omega-\tfrac{1}{2}\pi) - \sin k\,(\omega_o-\tfrac{1}{2}\pi)\} \; . \qquad \ldots\ldots \quad (193)$$

The complete perturbation $\delta\bar{\Omega}$ is made up from the contributions (190), (192) and (193). The presence of the second term in the square brackets of (193) explains, for the case $\ell = 3$ and $k = 1$, an apparent discrepancy between formulae of Merson[1] and Kozai[12]; Refs.1 and 12 differ in having factors $(1-15f/4)$ and $(1-5f/4)$ respectively but the discrepancy, noted by A. H. Cook[7], is now seen not to be a real one. Full agreement between Refs.1 and 12 has been demonstrated in a further paper by Merson[35].

It is observed that one unfortunate result follows from the separation of the J_ℓ perturbation into secular and long-period components: near the critical inclinations each component becomes very large, though the combination remains finite (and continuous) always. Thus if $\dot{\omega}$ is near to zero the separation should not be made.

The perturbation, $\delta\bar{\omega}$, of ω may be dealt with in the same way. If $I_{\ell k}$ is such that

$$I_{\ell k}(e,i) \cos k\,(\omega-\tfrac{1}{2}\pi) \;=\; \omega_{\ell k} \quad ,$$

then $\delta\bar{\omega}$ has

$$n\,J_\ell \left(\frac{R}{p}\right)^\ell I_{\ell o}(e,i)\,t \qquad (194)$$

arising for each $\ell > 2$ and $k = 0$. Terms with $k > 0$ may be combined with the main J_2 secular term and represented by a secular term

$$3J_2\,n'\left(\frac{R}{p_o}\right)^2 \left\{\left(1-\frac{5}{4}f_o\right) + \tfrac{1}{2}\,e(1-e^2)^{-1}\,(13-15f) \sum_{\ell,k} \frac{nJ_\ell}{k\dot{\omega}} \left(\frac{R}{p}\right)^\ell G_{\ell k}(e,i)\cos k\,(\omega_o-\tfrac{1}{2}\pi)\right\} t \qquad \ldots\ldots \quad (195)$$

together with long-period terms

$$\sum_{\ell,k} \frac{nJ_\ell}{k\dot{\omega}} \left(\frac{R}{p}\right)^\ell \left[I_{\ell k}(e,i) - \frac{e(1-e^2)^{-1}\,(13-15f)}{2k\left(1-\frac{5}{4}f\right)}\,G_{\ell k}(e,i)\right] \{\sin k\,(\omega-\tfrac{1}{2}\pi) - \sin k\,(\omega_o-\tfrac{1}{2}\pi)\} \; . \qquad \ldots\ldots \quad (196)$$

Similarly, if $K_{\ell k}(e,i)$ is such that

$$K_{\ell k}(e,i)\cos k(\omega-\tfrac{1}{2}\pi) = \sigma_{\ell k} \quad ,$$

then the perturbation $\delta\bar{\sigma}$ has

$$n\,J_\ell\left(\frac{R}{p}\right)^{\ell} K_{\ell o}(e,i)\,t \tag{197}$$

arising for each $\ell > 2$ and $k = 0$. The combination of terms for which $k > 0$ with the main J_2 secular term may be represented by a secular term

$$\frac{3}{2}J_2 n'\left(\frac{R}{p_o}\right)^2 (1-e_o^2)^{\frac{1}{2}}\left\{\left(1-\frac{3}{2}f_o\right)+2e(1-e^2)^{-1}\sum_{\ell,k}\frac{J_\ell}{kJ_2}\left(\frac{R}{p}\right)^{\ell-2}G_{\ell k}(e,i)\cos k(\omega_o-\tfrac{1}{2}\pi)\right\}t \quad \ldots\ldots \tag{198}$$

(with J_2 instead of $\dot{\omega}$ in the denominator of the summed terms since a factor $(1-5f/4)$ has been cancelled), together with long-period terms

$$\sum_{\ell,k}\frac{nJ_\ell}{k\dot{\omega}}\left(\frac{R}{p}\right)^{\ell}\left[K_{\ell k}(e,i)-\frac{3e(1-e^2)^{-\frac{1}{2}}}{k}G_{\ell k}(e,i)\right]\{\sin k(\omega-\tfrac{1}{2}\pi)-\sin k(\omega_o-\tfrac{1}{2}\pi)\} \quad . \quad \ldots\ldots \tag{199}$$

But, as with the short-period terms listed in section 10.1, it is the perturbation in M, rather than that in σ, which is important. To obtain this we follow the method of section 7, where n' was introduced by equation (63). Thus it follows from equation (8) that

$$\frac{d\bar{M}}{dt} = n' + \frac{1}{T_\omega}\int_0^{T_\omega}(n-n')dt + \frac{d\bar{\sigma}}{dt} \quad , \tag{200}$$

where

$$\int_0^{T_\omega}(n-n')dt = -2\pi\sum_{\ell}J_\ell\left(\frac{R}{p}\right)^{\ell}\frac{3\rho_\ell}{2(\ell+1)}$$

using equations (63) and (68). If $L_{\ell k}(e,i)$ is defined such that

$$L_{\ell k}(e,i)\cos k(\omega-\tfrac{1}{2}\pi) = -\frac{3\rho_{\ell k}}{2(\ell+1)}$$

then, since $K_{20}(e,i) + L_{20}(e,i) = 0$, the complication of terms in $G_{\ell k}(e,i)$ does not arise. The perturbation $\delta\bar{M}$ consists of the secular term

$$n' J_\ell \left(\frac{R}{p}\right)^\ell \{K_{\ell o}(e,i) + L_{\ell o}(e,i)\} t \tag{201}$$

for each $\ell > 0$ and $k = 0$, the main secular term

$$n't \quad , \tag{202}$$

where n' is absolutely constant, and long-period terms

$$\sum_{\ell,k} \frac{nJ_\ell}{k\dot{\omega}} \left(\frac{R}{p}\right)^\ell \{K_{\ell k}(e,i) + L_{\ell k}(e,i)\} \{\sin k (\omega - \tfrac{1}{2}\pi) - \sin k (\omega_o - \tfrac{1}{2}\pi)\} \quad . \tag{203}$$

We now outline the similar analysis which may be carried out to obtain secular and long-period perturbations associated with J_2^2. Here, as has already been mentioned, the expressions obtained depend on the point of the orbit with which the orbital elements are associated. If the smoothed elements of Merson[1] or Kozai[12] are used the long-period perturbation for semi-major axis turns out to be zero. For osculating elements defined at the node or perigee this is not the case. Here we give perturbations for elements at perigee, for which the second-order changes over one revolution have been given by equations (181) to (186).

From equations (181) to (183) the long-period perturbations of a, e and i are given by

$$\delta a_\omega = \frac{9 n' J_2^2}{2\dot{\omega}} \frac{R^4 a^2}{p^5} (1+e)^3 f \left(1 - \frac{5}{4}f\right) (\cos 2\omega - \cos 2\omega_o) \quad , \tag{204}$$

$$\delta e_\omega = \frac{3 n' J_2^2}{4\dot{\omega}} \left(\frac{R}{p}\right)^4 (1+e) f \left\{\left(1 - \frac{5}{4}f\right)(5+7e) + e(1-e)\left(\frac{7}{8} - \frac{15}{16}f\right)\right\} (\cos 2\omega - \cos 2\omega_o)$$

.... (205)

and

$$\delta i_\omega = \frac{3 n' J_2^2}{8\dot{\omega}} \left(\frac{R}{p}\right)^4 \sin 2i \left\{\left(1 - \frac{5}{4}f\right)(3+4e) - \left(\frac{7}{8} - \frac{15}{16}f\right)e^2\right\} (\cos 2\omega - \cos 2\omega_o) \quad ,$$

.... (206)

C

respectively. In the above equations n' is used since (with $\sigma_2 - \frac{1}{2}\rho_2 = 0$ from section 7)

$$\frac{2\pi}{T_\omega} = n' + O(J_2^2, J_3 \ldots\ldots) \quad ;$$

the distinction becomes vital for the perturbations $\delta\Omega_\omega$, $\delta\omega_\omega$ and $\delta\sigma_\omega$ given below. Zero suffixes refer to initial perigee values.

For $\delta\Omega_\omega$ the procedure is similar to that for the J_ℓ perturbations. Combining the J_2^2 terms with the main J_2 secular term it may be shown that the total perturbation is representable by a secular term

$$-\frac{3}{2} n' J_2 \left(\frac{R}{p_o}\right)^2 \cos i_o \left[1 + \frac{1}{4} J_2 \left(\frac{R}{p}\right)^2 \left\{(9-10f) + \tfrac{1}{4}e^2 (4+5f) + 5f \left(3+4e - \frac{14-15f}{4(4-5f)} e^2\right) \times \cos 2\omega_o\right\}\right] t \;, \tag{207}$$

together with a long-period term

$$\frac{3 n' J_2^2}{16 \dot{\omega}} \left(\frac{R}{p}\right)^4 \cos i \left[(3+4e)\ (4-5f) - \frac{e^2(56-120f+75f^2)}{4(4-5f)}\right] (\sin 2\omega - \sin 2\omega_o) \;. \tag{208}$$

Similarly, the combination of $\delta\omega_\omega$ with the main J_2 secular term may be represented by a secular term

$$3 n' J_2 \left(\frac{R}{p_o}\right)^2 \left[\left(1 - \frac{5}{4} f_o\right) + \frac{1}{128} J_2 \left(\frac{R}{p}\right)^2 \left\{2(192-412f+215f^2) + e^2(56-36f-45f^2) + 16f(13-15f) \left(3+4e - \frac{14-15f}{4(4-5f)} e^2\right) \cos 2\omega_o\right\}\right] t \quad \ldots\ldots \tag{209}$$

together with a long-period term

$$\frac{3 n' J_2^2}{128 \dot{\omega}} \left(\frac{R}{p}\right)^4 \left[16e^{-1} (4-5f)\ (f-2e^2+2e^2 f) - 2(48-166f+135f^2) + \frac{e^2(112-746f+900f^2-175f^3)}{4-5f}\right] (\sin 2\omega - \sin 2\omega_o) \;. \tag{210}$$

For the final element, σ, the combination of $\delta\sigma_\omega$ with the main J_2 secular term may be represented by a secular term

$$\frac{3}{2}\, n'\, J_2 \left(\frac{R}{p_o}\right)^2 (1-e_o^2)^{\frac{1}{2}} \Bigg[\left(1-\frac{3}{2}\, f_o\right) + \frac{1}{32}\, J_2 \left(\frac{R}{p}\right)^2 (1-e^2)^{-1}$$
$$\times \Big\{2(48-116f+67f^2) + 12e(2-3f)^2 - 2e^2(16-52f+59f^2)$$
$$+ 4e^3(2-3f)^2 - e^4\left(35-41f-\frac{79}{8}\, f^2\right)$$
$$+ 3f(1-e^{\cdot}) \left(4(4-5f)(3+4e) - e^2(14-15f)\right) \cos 2\omega_o\Big\}\Bigg] t \qquad \ldots\ldots (211)$$

together with a long-period term

$$-\frac{3\, n'\, J_2^2}{128\,\dot{\omega}} \left(\frac{R}{p}\right)^4 f\, (1-e^2)^{-\frac{1}{2}} \Big[\big\{16e^{-1}(4-5f) - 2(98-123f) - 4e(98-127f) + e^2(206-231f)$$
$$+ 4e^3(58-71f) - 2e^4(32-33f)\big\}\, (\sin 2\omega - \sin 2\omega_o)$$
$$+ \frac{9}{32}\, e^4 f\, (\sin 4\omega - \sin 4\omega_o)\Big] \quad . \qquad (212)$$

Again, however, it is the perturbation in M, rather than that in σ, which is important. This may be obtained by starting from equation (200) where now, introducing the second-order term into equation (63),

$$\int_0^{T_\omega} (n-n')\,dt \;=\; -\tfrac{1}{2}\,\Delta\rho - \frac{3}{2}\int_0^{T_\omega} na^2\, \frac{U^2}{\mu^2}\, dt \quad .$$

Since $\frac{1}{2}\rho_2 = \sigma_2$ it follows that we may write

$$\frac{d\bar{M}}{dt} = n' + n'\, J_2^2 \left(\frac{R}{p}\right)^4 (\sigma_{22} - \tfrac{1}{2}\,\rho_{22} - \nu_{22}) \qquad (213)$$

where σ_{22}, ρ_{22} and ν_{22} are defined as follows: σ_{22} is such that

$\Delta\sigma = 2\pi\, J_2^2 \left(\frac{R}{p}\right)^4 \sigma_{22}$ for $\Delta\sigma$ given by equation (186); ρ_{22} is given similarly by

$\Delta\rho$ where, extending the set of equations (181) to (186),

$$\Delta\rho = \frac{3\pi}{16}\, J_2^2 \left(\frac{R}{p}\right)^4 (1-e^2)^{-\frac{1}{2}} \Big[2(48-108f+51f^2) + 12e(2-3f)^2 - e^2(72-192f+189f^2)$$
$$+ 4e^3(2-3f)^2 - 3e^4\left(10-14f+\frac{7}{4}\, f^2\right)$$
$$+ f\big\{12(26-33f) + 12e(38-49f) - 6e^2(62-75f)$$
$$- 12e^3(30-37f) + 72e^4(1-f)\big\} \cos 2\omega - \frac{9}{8}\, e^4 f^2 \cos 4\omega\Big] ; \qquad \ldots\ldots (214)$$

ν_{22} is similar, with

$$\Delta\nu = \frac{3}{2}\int_0^{T_\omega} na^2 \frac{U^2}{\mu^2} dt ,$$

and it follows from Merson's work[1] that

$$\Delta\nu = \frac{3\pi}{4} J_2^2 \left(\frac{R}{p}\right)^4 (1-e^2)^{-\frac{1}{2}} \left[\left(1 + 3e^2 + \frac{3}{8} e^4\right)\left(1 - 3f + \frac{27}{8} f^2\right) + \frac{9}{2}\left(1 + \frac{1}{6} e^2\right)\left(1 - \frac{3}{2} f\right) f e^2 \cos 2\omega + \frac{9}{128} f^2 e^4 \cos 4\omega\right] . \quad (215)$$

Hence, after some simplification,

$$\frac{d\bar{M}}{dt} = n'\left[1 - \frac{3}{64} J_2^2 \left(\frac{R}{p}\right)^4 (1-e^2)^{3/2} \{(8-8f-5f^2) + 2f(8e^{-1}(4-5f) - (14-15f))\cos 2\omega\}\right].$$

Thus we obtain, for M_ω, a secular term

$$n'\left\{1 - \frac{3}{64} J_2^2 \left(\frac{R}{p}\right)^4 (1-e^2)^{3/2} (8-8f-5f^2)\right\} t \quad (216)$$

and a long-period term

$$- \frac{3\, n'\, J_2^2}{64\, \dot{\omega}} \left(\frac{R}{p}\right)^4 f\, (1-e^2)^{3/2} \{8e^{-1}(4-5f) - (14-15f)\} (\sin 2\omega - \sin 2\omega_o) . \quad (217)$$

It is now desirable to summarise the results of this sub-section, showing how the formulae lead to the possibility of an algorithm for computing the position in space, at a given time, of an earth satellite which is subject only to earth asphericity perturbations. We suppose that the position and velocity of the satellite at a given time, t_1, are known and that its position at time t_2 is required. A possible sequence of steps, in outline only, would be as follows:

(i) obtain the osculating elements, a_1, e_1, i_1, Ω_1, ω_1 and M_1 ($= \sigma_1 = \chi_1$) at the epoch t_1, by standard formulae[29]; also the true anomaly, v_1;

(ii) from equation (10) obtain U, given r_1 and β_1 at t_1, and hence obtain a' from equation (62) and n' from $n'^2 a'^3 = \mu$;

(iii) obtain the initial purely short-period terms, relative to elements at perigee, from equations (172) to (176) and (179) with secular terms omitted (v replaced by $v-M$ in $\delta\Omega$ and $\delta\omega$, and $n't$ dropped from δM) and the appropriate $F(e,i,\omega)$ terms substituted; remove these short-period terms, $\delta_1 a$ etc, from $a_1, \ldots, M_1$ to give a_o $(= a_1 - \delta_1 a)$ etc;

(iv) obtain $\dot{\omega} = 3 J_2 n' \left(\frac{R}{p_o}\right)^2 \left(1 - \frac{5}{4}\sin^2 i_o\right)$; also the functions of p_o, e_o and i_o which are required for secular and long-period perturbations;

(v) obtain ω', where $\omega' = \omega_1 + \dot{\omega}(t_2 - t_1)$;

(vi) obtain the combined secular and long-period perturbations $\delta_2 a, \ldots \delta_2 M$; for example $\delta_2\Omega$ is given by contributions from equations (190), (192), (193), (207) and (208), with $p, e, i, \omega_o, \omega$ and t replaced by p_o, e_o, i_o, ω_1, ω' and $t_2 - t_1$ respectively; the duplicated main term in equations (192) and (207) is, of course, taken once only;

(vii) obtain $a_3, \ldots, M_3$ from $a_3 = a_o + \delta_2 a$ etc; hence also v_3 (using Kepler's equation);

(viii) as in step (iii) obtain short-period perturbations $\delta_3 a$ etc, based on a_3 etc; hence obtain $a_2, \ldots, M_2$ from $a_2 = a_3 + \delta_3 a$ etc;

(ix) obtain position from the osculating elements a_2, e_2, i_2, Ω_2, ω_2 and M_2.

Steps (i) to (iv) are independent of t_2 and would be performed once only if position were required at several times t_2. No difficulties occur near the two critical inclinations if the formulae are handled correctly, but the usual problems arise[12,21] for the singularities in the elements at $e = 0$, $i = 0$ and $i = \pi$.

10.3 Secular perturbations associated with a disturbing body

It is supposed that the orbit of the satellite is close enough to the earth for the secular variations in Ω and ω due to the earth's oblateness to be still the dominant effects. Rates of change of satellite elements due to the principal terms of the disturbing potential of a distant body have been given by equations (136) to (160), averaged with respect to the mean anomalies of both the satellite and the disturbing body.

To obtain perturbations over a period which is long in comparison with the periods of the satellite and the disturbing body it is necessary, as in section 9.3, to consider the secular variation of ω. Due to the presence of Ω in the equations it is now also necessary to consider the secular variation of this element. Following the example of Smith[25], we do not give long-period

variations, which are liable to become rather complicated, but only the secular variations obtained by averaging equations (136) to (160) with respect to Ω and ω.

Since Ω only occurs as part of the trigonometrical argument $(\Omega-\Omega_d)$ it is actually with respect to this quantity that the first averaging takes place. To avoid, for n = -5 and k = 0, terms containing factors $\sin 2\omega_d$ and $\cos 2\omega_d$ we also, in effect, average with respect to ω_d; these terms, containing also the factor e_d^2, are in any case negligible.

Since averaging with respect to ω is being performed secular perturbations associated with any $k \neq 0$ must vanish. Of the (n,k) values considered in section 8 this leaves us with (-3,0 and (-5,0) only. For the first case, equations (136) to (140) lead easily to

$$\frac{d\,e_{-3,0}}{dt} = \frac{d\,i_{-3,0}}{dt} = 0 \quad , \tag{218}$$

$$\frac{d\,\Omega_{-3,0}}{dt} = -\frac{3\mu_d}{4\,a_d^3\,n}(1-e_d^2)^{-3/2}(1-e^2)^{-\frac{1}{2}}\left(1+\frac{3}{2}e^2\right) \times\left(1-\frac{3}{2}\sin^2 i_d\right)\cos i \quad , \tag{219}$$

$$\frac{d\,\omega_{-3,0}}{dt} + \cos i\,\frac{d\,\Omega_{-3,0}}{d\,t} = \frac{3\mu_d}{4\,a_d^3\,n}(1-e_d^2)^{-3/2}(1-e^2)^{\frac{1}{2}}\left(1-\frac{3}{2}\sin^2 i_d\right)\left(1-\frac{3}{2}\sin^2 i\right) \quad \ldots\ldots \tag{220}$$

and

$$\frac{d\,\sigma_{-3,0}}{dt} = -\frac{\mu_d}{4\,a_d^3\,n}(1-e_d^2)^{-3/2}(7+3e^2)\left(1-\frac{3}{2}\sin^2 i_d\right) \times\left(1-\frac{3}{2}\sin^2 i\right) . \tag{221}$$

The secular results given by Kozai[26] follow at once from equations (219) and (220), to the order in e to which he works. We recall that the inclination i_d is relative to the earth's equator and is not the (almost constant) inclination relative to the ecliptic.

For the other case, equations (156) to (160) lead to

$$\frac{d\,e_{-5,0}}{dt} = \frac{d\,i_{-5,0}}{dt} = 0 \quad , \tag{222}$$

$$\frac{d\,\Omega_{-5,0}}{dt} = \frac{-45\mu_d}{1024\,a_d^5\,n}\,a^2(1-e_d^2)^{-7/2}\left(1+\frac{3}{2}e_d^2\right)(1-e^2)^{-\frac{1}{2}}\left(1+5e^2+\frac{15}{8}e^4\right)$$
$$\times\,(8-40\sin^2 i_d+35\sin^4 i_d)(4-7\sin^2 i)\cos i \quad , \tag{223}$$

$$\frac{d\,\omega_{-5,0}}{dt} + \cos i\,\frac{d\,\Omega_{-5,0}}{dt} = \frac{45\mu_d}{2048\,a_d^5\,n}\,a^2(1-e_d^2)^{-7/2}\left(1+\frac{3}{2}e_d^2\right)(1-e^2)^{\frac{1}{2}}\left(1+\frac{3}{4}e^2\right)$$
$$\times\,(8-40\sin^2 i_d+35\sin^4 i_d)\,(8-40\sin^2 i+35\sin^4 i) \quad \ldots\ldots \tag{224}$$

and

$$\frac{d\,\sigma_{-5,0}}{dt} = \frac{-81\mu_d}{2048\,a_d^5\,n}\,a^2(1-e_d^2)^{-7/2}\left(1+\frac{3}{2}e_d^2\right)\left(1+\frac{25}{12}e^2+\frac{5}{12}e^4\right)$$
$$\times\,(8-40\sin^2 i_d+35\sin^4 i_d)\,(8-40\sin^2 i+35\sin^4 i) \quad . \tag{225}$$

Smith[25] gives expressions containing terms covered by our equations (222) to (224). These terms in Ref.25 contain a number of errors, however, as has been confirmed by Smith in the private communication mentioned in section 8.

It should be noted just why the assumption had to be made at the beginning of this section that the perturbations due to oblateness were still dominant. The interaction of these perturbations on those arising from the disturbing body has been considered, but the interaction of the latter on the former has been disregarded. Thus to analyse the complete effect of the mutual interaction on i and Ω it is necessary, for first-order results, to integrate the simultaneous differential equations for di/dt and dΩ/dt given by equations (137) and (138) with the oblateness terms added on the right hand sides. Analysis on these lines has been carried out by Allan and Cook[36].

The above remarks make the higher-order terms given by equations (222) to (225) rather irrelevant in the case of luni-solar perturbations of an earth

satellite; for if a/a_d were small they would be very small and otherwise the treatment would be invalid anyway. These terms would only have been useful if the earth had been much more oblate or if the moon had been lighter and in a closer orbit.

A final remark which is worth making in connection with perturbations of a satellite due to a disturbing body concerns the perturbation of the mean anomaly. As in section 10.2 it is the perturbation of M rather than that in σ which is really required. Returning to the case of the disturbing body regarded as fixed during one revolution of the satellite (section 8.3), the mean rate of change of M is given, to first order, by

$$\frac{d\bar{M}}{dt} = n' + \frac{n'}{2\pi}\left(\Delta\sigma + \frac{3}{4}\Delta\rho\right) ,$$

where both $\Delta\sigma$ and $\Delta\rho$ are given by equation (94) of which the left hand side actually <u>is</u> $\Delta\rho$. However, the situation is more complicated when allowance is made for movement of the disturbing body, since the equation of energy no longer leads, as in section 7, to a constant n'.

11 CONCLUSIONS

The disturbing function for an axi-symmetric potential field has been written as the doubly infinite Legendre series

$$-\sum_{n=-\infty}^{\infty} \mu J_n R^n r^{-n-1} P_\ell(\sin\beta) ,$$

where r and $\frac{1}{2}\pi-\beta$ are polar co-ordinates, $\ell = n$ if $n > 0$, and $\ell = -n-1$ if $n < 0$. For the field of the earth, assumed axi-symmetric, the J_n, for $n > 2$, are the now standard coefficients, R being the mean equatorial radius of the earth; J_1, J_0 and J_{-1} have simple meanings and may easily be made to vanish; the series with the J_n for $n < -2$ only may be applied to the study of luni-solar perturbations of an earth satellite, if the J_n and R are properly interpreted.

The effect of the general term, U_n, of the series on the motion of a satellite in the field has been obtained by first expressing U_n in terms of the usual orbital elements and mean anomaly, and then averaging with respect to the mean anomaly to eliminate short-period terms. The average U_n becomes the sum of a series of terms in $\cos k(\omega-\frac{1}{2}\pi)$, with $0 < k < \ell$. For the general such term, $U_{nk}(av)$, the changes in the orbital elements over a complete revolution of the satellite have been found by making use of Lagrange's planetary equations.

Formulae for the secular and long-period motion of a satellite in the gravitational field of the earth have been developed. The long-period perturbations arise from the secular motion of perigee associated with J_2. At the two critical inclinations this motion ceases to exist, but the formulae for secular and long-period motion of each element have been developed in such a way that for the combined perturbation there is no singularity near the critical inclinations.

In all sets of formulae for the perturbation of orbital elements the perturbation of the sixth element, omitted by many authors, has been included. By use of the constant mean motion, n', of Merson the formula for the long-period perturbation of mean anomaly has the same accuracy as the formulae for the other five orbital elements. It should be remarked, however, that for an actual close earth satellite the long-period variation of mean anomaly is essentially governed - in a largely unpredictable fashion - by atmospheric drag when, in addition, n' is no longer constant.

Application to the study of luni-solar perturbations has required consideration of more than one axis of symmetry. Formulae for the perturbations of elements defined in terms of a pseudo-equator have accordingly been transformed into formulae in the normal elements. The plane perpendicular to the direction of a disturbing body is one such pseudo-equator and the orbital plane of the body another.

Appendix A

LEGENDRE EXPANSIONS FOR GRAVITATIONAL POTENTIAL

Consider first the simple case of Fig. 1(a) in which P' represents the location of an isolated point mass m'. P is a general point and O is an arbitrary origin of polar co-ordinates.

The potential at P due to P' is m'/P'P. Taking the axis of polar co-ordinates along OP' we have

$$\frac{m'}{P'P} = \frac{m'}{(r^2 + r'^2 - 2rr'\cos\theta)^{\frac{1}{2}}} \quad . \tag{A1}$$

There are two cases to consider according to whether $r > r'$ or $r < r'$.

If $r > r'$, the expansion of $m'r^{-1}\{1-2(r'/r)\cos\theta + (r'/r)^2\}^{-\frac{1}{2}}$ as an infinite series in powers of r'/r is valid. As is remarked at the beginning of Appendix B, this expansion may be used to define the Legendre polynomials $P_\ell(\cos\theta)$ and in fact

$$\frac{m'}{P'P} = m' \sum_{\ell=0}^{\infty} \frac{r'^\ell}{r^{\ell+1}} P_\ell\,(\cos\theta) \quad . \tag{A2}$$

If $r < r'$, we have similarly

$$\frac{m'}{P'P} = m' \sum_{\ell=0}^{\infty} \frac{r^\ell}{r'^{\ell+1}} P_\ell\,(\cos\theta) \quad . \tag{A3}$$

Next consider an extension to the case of several point masses; in Fig. 1(b) there are three. O is still an arbitrary origin and the polar axis OZ is now also chosen arbitrarily; OX is an arbitrary line of zero azimuth in the plane perpendicular to OZ.

The potential at P due to m' at P' may be expressed as a series in the $P_\ell\,(\cos\varphi)$. In order to add this potential to that due to the other masses it is desirable to rewrite the series, expressing it in terms of the angles θ and θ'. We note that, if λ and λ' are the azimuths of OP and OP' respectively,

$$\cos\varphi = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\lambda-\lambda') \quad . \tag{A4}$$

Hence the addition theorem proved in Appendix C may be applied and, from equation (C1), using the associated Legendre functions defined in Appendix B,

$$P_\ell(\cos\varphi) = \sum_{m=0}^{\ell} u_m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta') P_\ell^m(\cos\theta) \cos m(\lambda-\lambda') \quad , \qquad (A5)$$

where $u_m = 1$ if $m = 0$ but 2 if $m > 0$.

For the case $r > r'$ we now have, from equations (A2) and (A5),

$$\text{potential (due to } m') = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} r^{-\ell-1} P_\ell^m(\cos\theta) \left(C_\ell^m(m') \cos m\lambda + S_\ell^m(m') \sin m\lambda\right) \quad \ldots\ldots \quad (A6)$$

where

$$C_\ell^m(m'),\ S_\ell^m(m') = m' \, u_m \, r'^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta') \, (\cos m\lambda', \sin m\lambda').$$

The double series is convergent and must remain so on combining with equivalent series for other point masses, m'', m''' etc, so long as $r > r''$, $r > r'''$ etc. The potential due to a finite set of masses is simply that given by equation (A6) with $C_\ell^m(m')$ and $S_\ell^m(m')$ replaced by $C_\ell^m(m') + C_\ell^m(m'') + \ldots$ and $S_\ell^m(m') + S_\ell^m(m'') + \ldots$ respectively.

When we consider the potential at P due to an actual gravitating body, such as the earth, there is no difficulty with the convergence on passing from a sum to an integral, so long as $r' < r$ for all points P' of the body - in any case it is arguable whether a finite body contains a finite or infinite number of points. Thus for any given distribution of matter the general expression for the potential at any point outside a sphere containing all the matter is given by

$$U' = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} r^{-\ell-1} P_\ell^m(\cos\theta) \left(C_\ell^m \cos m\lambda + S_\ell^m \sin m\lambda\right) \quad , \qquad (A7)$$

where

$$C_\ell^m,\ S_\ell^m = \int_{(\text{matter})} u_m \, r'^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta') \, (\cos m\lambda', \sin m\lambda') dm' \quad .$$

The expression, by equation (9) of section 4.1, for the earth's gravitational potential now follows with minor changes of notation.

By consideration of the case r < r' we conclude, similarly, that if a given distribution of matter occurs entirely outside a certain sphere, then the potential at all points inside this sphere is expressible by

$$U' = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} r^{\ell} P_{\ell}^{m}(\cos\theta)\,(\bar{C}_{\ell}^{m} \cos m\lambda + \bar{S}_{\ell}^{m} \sin m\lambda) \quad , \qquad (A8)$$

where

$$\bar{C}_{\ell}^{m}, \bar{S}_{\ell}^{m} = \int_{(\text{matter})} u_m\, r'^{-\ell-1} \frac{(\ell-m)!}{(\ell+m)!} P_{\ell}^{m}(\cos\theta')\,(\cos m\lambda', \sin m\lambda')\, dm' \quad .$$

The results expressed by equations (A7) and (A8) may be combined into a single equation, the validity of which holds throughout any empty region bounded by two spheres. We take n to range over all integers, positive and negative, defining $\ell = n$ if $n > 0$ and $\ell = -n-1$ if $n < 0$. The equation is

$$U' = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\ell} r^{-n-1} P_{\ell}^{m}(\cos\theta)\,(C_{n}^{m} \cos m\lambda + S_{n}^{m} \sin m\lambda) \quad , \qquad (A9)$$

where

$$C_{n}^{m}, S_{n}^{m} = \bar{C}_{\ell}^{m}, \bar{S}_{\ell}^{m} \quad \text{if} \quad n < 0 \quad .$$

Absence of matter from the spherically bounded region is a sufficient condition for the convergence of the series but not, in general, a necessary one. For evidence that it may not be necessary section 24.06 of Ref.37 should be consulted. For a case when it is necessary, see Fig.1(c). Matter with unit line density is assumed to be distributed between A and B, points for both of which $\theta = 0$ and for which $r = a$ and b respectively. For $r > b$, the potential is

$$\sum_{\ell=0}^{\infty} \frac{b^{\ell+1} - a^{\ell+1}}{(\ell+1)\, r^{\ell+1}} P_{\ell}(\cos\theta) \quad ,$$

which at a point P for which $\theta = \pi$ gives

$$\log \frac{b+r}{a+r} \quad .$$

For $r < a$ there is a similar expression in positive powers of r, giving the same sum. For $a < r < b$ the potential, still along $\theta = \pi$, is of course given by the same logarithmic function, but now it has no expansion in powers of r.

Appendix B

LEGENDRE FUNCTIONS

The basic functions are the Legendre polynomials $P_n(x)$ which, as usual in mathematics, may be defined in many different ways. One definition of $P_n(x)$ is as the coefficient of y^n in the formal power-series expansion of the function $(1-2\,xy+y^2)^{-\frac{1}{2}}$. From this definition it is straightforward[38] to obtain the explicit formula for $P_n(x)$ given by Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n \quad . \tag{B1}$$

This formula, of course, provides another possible definition of $P_n(x)$.

The first six polynomials are given by:-

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2-1),$$

$$P_3(x) = \tfrac{1}{2}(5x^3-3x), \qquad P_4(x) = \tfrac{1}{8}(35x^4-30x^2+3),$$

$$P_5(x) = \tfrac{1}{8}(63x^5-70x^3+15x) \quad .$$

The associated Legendre function $P_n^m(x)$ provides a generalization of $P_n(x)$ to which it reduces when $m = 0$. It may be defined by

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \quad , \tag{B2}$$

so that, in view of Rodrigues' formula, (B1),

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n\, n!}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n \quad . \tag{B3}$$

Some warning remarks on the above definition are in order. For although there is universal agreement on the definition of $P_n(x)$, authorities differ over $P_n^m(x)$.

Jeffreys and Jeffreys[37] use equation (B2) and this definition is incorporated in the 1961 recommendation of the International Astronomical Union[28]. MacRobert[38], however, and some older texts write $P_n^m(x) = (x^2-1)^{m/2}\, d^m\, P_n(x)/dx^m$. This definition is obviously more profitable if $|x| > 1$, but less profitable if $|x| < 1$, x being real.

Groves[3] and Merson[6] have tried for the best of both worlds, following MacRobert with $P_n^m(x)$ and introducing $T_n^m(x)$, in the notation of Ferrers[39], as a function identical with our $P_n^m(x)$. The present author feels that this double usage is unnecessary, contrary to the I.A.U. recommendation and confusing. A possible source of danger is illustrated in Ref.3, which introduces $T_n^m(x)$ first and then, when $|x| > 1$, uses MacRobert's $P_n^m(x)$, writing $P_n^m(x) = (-1)^{\frac{1}{2}m}\, T_n^m(x)$; but this last relation is wrong, since when x is real and $|x| > 1$, the correct principal-value relation is $(1-x^2)^{m/2} = (-1)^{m/2}\,(x^2-1)^{m/2}$, leading to $P_n^m(x) = (-1^{3m/2}\, T_n^m(x)$.

A further point of divergence in the definition of the associated Legendre functions does not concern us in this paper but is mentioned for completeness. It may be convenient to multiply $P_n^m(x)$ by an arithmetic factor and to use the resulting function, denoted say by $p_n^m(x)$, instead of $P_n^m(x)$. Jeffreys and Jeffreys[37] take, as this arithmetic factor, $(n-m)!/n!$ but the I.A.U. recommend $\{(n-m)!/(n+m)!\}^{\frac{1}{2}}$. The object of the factor is to achieve some measure of normalization and Kaula[40] points out that for true normalization the factor $\{u_m(n-m)!\,(2n+1)/(n+m)!\}^{\frac{1}{2}}$, where u_m is 1 for m = 0 and 2 for $m \neq 0$, should be used. Kaula's factor is becoming widely used in evaluating the earth's tesseral harmonic coefficients[41], but it has one major weakness: the factor is not unity when m = 0, so that the basic Legendre polynomials themselves are affected.

If $x = \cos\varphi$, where $0 < \varphi < \pi$, the functions $P_n^m(x)$ may be expressed rather simply. For values of n up to 3 the functions, excluding Legendre polynomials (m = 0) already given, are as follows:

$$P_1^1(x) = \sin\varphi, \qquad P_2^1(x) = 3\sin\varphi\cos\varphi, \qquad P_2^2(x) = 3\sin^2\varphi,$$

$$P_3^1(x) = \tfrac{3}{2}\sin\varphi\,(5\cos^2\varphi-1), \quad P_3^2 = 15\sin^2\varphi\cos\varphi, \quad P_3^3(x) = 15\sin^3\varphi\ .$$

Appendix C

THE ADDITION THEOREM FOR LEGENDRE FUNCTIONS

The addition theorem is the key to the expressing of Legendre polynomials relating to one axis in terms of Legendre functions relating to another axis. Thus in Fig.1(b) we suppose that OZ and OP' are particular (given) axes and that OP is a general direction; OP makes angles θ and φ with OZ and OP' respectively, the angle between these axes being θ'. Then the theorem states that Legendre polynomials in $\cos\varphi$ (i.e. relating to OP') are given in terms of Legendre functions in $\cos\theta$ (i.e. relative to OZ) by the expressions

$$P_n(\cos\varphi) = \sum_{m=0}^{n} u_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta')\, P_n^m(\cos\theta)\cos m\lambda \quad , \qquad (C1)$$

where λ is the (azimuth) angle between the projections of OP and OP' on the plane perpendicular to OZ, $u_m = 1$ if $m = 0$ but 2 if $m > 0$, and $n > 0$.

Proof of the addition theorem is usually given[37,38] as part of a complete mathematical treatment of spherical harmonics. It is possible, however, to prove it by elementary algebra, and we proceed to do this.

The theorem is trivial when $n = 0$; when $n = 1$ it reduces to

$$\cos\varphi = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos\lambda \qquad (C2)$$

which is true by spherical trigonometry. We can, in fact, use this last expression to give a purely algebraic statement of the theorem: if φ is <u>defined</u> by equation (C2), then (C1) holds in general.

It is necessary to prove the theorem for $n \geq 2$ and this may be done by induction. Thus we suppose that, for an arbitrary $\ell \geq 2$, the theorem is true for $n = \ell-2$ and $n = \ell-1$. If its truth for $n = \ell$ can be established on the basis of this supposition, proof for all n will be complete.

From equation (D5) of Appendix D with $n = \ell-1$, $m = 0$ and $x = \cos\varphi$, from the induction hypothesis for $P_{\ell-1}(\cos\varphi)$ and $P_{\ell-2}(\cos\varphi)$, and from equation (C2), it follows that

$$P_\ell(\cos\varphi) = \frac{2\ell-1}{\ell}(\cos\theta'\cos\theta + \sin\theta'\sin\theta\cos\lambda)$$
$$\times \sum_{m=0}^{\ell-1} u_m \frac{(\ell-m-1)!}{(\ell+m-1)!} P^m_{\ell-1}(\cos\theta') P^m_{\ell-1}(\cos\theta)\cos m\lambda$$
$$- \frac{\ell-1}{\ell}\sum_{m=0}^{\ell-2} u_m \frac{(\ell-m-2)!}{(\ell+m-2)!} P^m_{\ell-2}(\cos\theta') P^m_{\ell-2}(\cos\theta)\cos m\lambda \quad . \tag{C3}$$

Thus, since

$$\cos\lambda\cos m\lambda = \tfrac{1}{2}(\cos\overline{m+1}\lambda + \cos\overline{m-1}\lambda) \quad , \tag{C4}$$

$P_\ell(\cos\varphi)$ is certainly expressible by

$$P_\ell(\cos\varphi) = \sum_{m=0}^{\ell} f(\theta', \theta, \ell, m)\cos m\lambda \quad , \tag{C5}$$

and it remains to be shown that

$$f(\theta', \theta, \ell, m) = u_m \frac{(\ell-m)!}{(\ell+m)!} P^m_\ell(\cos\theta') P^m_\ell(\cos\theta) \quad . \tag{C6}$$

The cases $m = \ell$ and $m = \ell-1$ are quite easy and are considered before the general case $(0 \leq m \leq \ell-2)$. We need - immediate from equation (B3) of Appendix B - the expressions

$$P^n_n(\cos\theta) = \frac{(2n)!\ \sin^n\theta}{2^n\ n!} \qquad \text{and} \qquad P^{n-1}_n(\cos\theta) = \frac{(2n)!\ \sin^{n-1}\theta\cos\theta}{2^n\ n!} \quad .$$

Using these expressions it may be seen very easily from equations (C3), (C4) and (C5) that

$$f(\theta', \theta, \ell, \ell) = \frac{2\ell-1}{\ell}\,\frac{\sin\theta'\sin\theta\, P^{\ell-1}_{\ell-1}(\cos\theta') P^{\ell-1}_{\ell-1}(\cos\theta)}{(2\ell-2)!}$$
$$= \frac{2\, P^\ell_\ell(\cos\theta') P^\ell_\ell(\cos\theta)}{(2\ell)!} \quad .$$

Similarly,

$$f(\theta', \theta, \ell, \ell-1) = \frac{2\ell-1}{\ell}\left\{\frac{2\cos\theta'\cos\theta\, P_{\ell-1}^{\ell-1}(\cos\theta')\, P_{\ell-1}^{\ell-1}(\cos\theta)}{(2\ell-2)!} + \frac{\sin\theta'\sin\theta\, P_{\ell-1}^{\ell-2}(\cos\theta')\, P_{\ell-1}^{\ell-2}(\cos\theta)}{(2\ell-3)!}\right\}$$

$$= \frac{2\, P_{\ell}^{\ell-1}(\cos\theta')\, P_{\ell}^{\ell-1}(\cos\theta)}{(2\ell-1)!} \quad .$$

For the general case, with $0 \leqslant m \leqslant \ell-2$, bearing in mind the special case when $m = 0$,

$$\begin{aligned} f(\theta', \theta, \ell, m) = \frac{2\ell-1}{\ell}\Big\{ & \cos\theta'\cos\theta\, u_m \frac{(\ell-m-1)!}{(\ell+m-1)!}\, P_{\ell-1}^{m}(\cos\theta')\, P_{\ell-1}^{m}(\cos\theta) \\ & + \sin\theta'\sin\theta\,(u_m-1)\frac{(\ell-m)!}{(\ell+m-2)!}\, P_{\ell-1}^{m-1}(\cos\theta')\, P_{\ell-1}^{m-1}(\cos\theta) \\ & + \tfrac{1}{2}\sin\theta'\sin\theta\, u_{m+1}\frac{(\ell-m-2)!}{(\ell+m)!}\, P_{\ell-1}^{m+1}(\cos\theta')\, P_{\ell-1}^{m+1}(\cos\theta)\Big\} \\ & - \frac{\ell-1}{\ell}\, u_m \frac{(\ell-m-2)!}{(\ell+m-2)!}\, P_{\ell-2}^{m}(\cos\theta')\, P_{\ell-2}^{m}(\cos\theta) \quad . \end{aligned}$$

Now $P_{\ell-1}^{m-1}$ may be eliminated by use of equation (D3) of Appendix D expressed in the appropriate form with $P_{\ell-1}^{m-1}$ on the R.H.S. Similarly, $P_{\ell-1}^{m+1}$ may be eliminated by use of equation (D6), with $n = \ell-1$. This leads to

$$\begin{aligned} f(\theta', \theta, \ell, m) = {} & g_1 \cos\theta'\cos\theta\, P_{\ell-1}^{m}(\cos\theta')\, P_{\ell-1}^{m}(\cos\theta) \\ & - g_2 \left\{\cos\theta'\, P_{\ell-1}^{m}(\cos\theta')\, P_{\ell-2}^{m}(\cos\theta) + \cos\theta\, P_{\ell-2}^{m}(\cos\theta')\, P_{\ell-1}^{m}(\cos\theta)\right\} \\ & + g_3\, P_{\ell-2}^{m}(\cos\theta')\, P_{\ell-2}^{m}(\cos\theta) \quad , \end{aligned}$$

where

$$g_1 = \frac{2\ell-1}{\ell}\left\{u_m \frac{(\ell-m-1)!}{(\ell+m-1)!} + \frac{u_m-1}{(\ell-m)^2}\frac{(\ell-m)!}{(\ell+m-2)!} + \tfrac{1}{2}u_{m+1}(\ell-m-1)^2\frac{(\ell-m-2)!}{(\ell+m)!}\right\}$$

$$= \frac{(2\ell-1)(\ell-m)!}{\ell(\ell-m)^2(\ell+m)!}\{u_m(\ell^2-m^2) + (u_m-1)(\ell+m)(\ell+m-1) + \tfrac{1}{2}u_{m+1}(\ell-m)(\ell-m-1)\}$$

$$= \frac{u_m(2\ell-1)^2}{(\ell-m)^2}\frac{(\ell-m)!}{(\ell+m)!} ,$$

$$g_2 = \frac{2\ell-1}{\ell}\left\{\frac{u_m-1}{(\ell-m)^2}\frac{(\ell-m)!}{(\ell+m-2)!} + \tfrac{1}{2}u_{m+1}(\ell+m-1)(\ell-m-1)\frac{(\ell-m-2)!}{(\ell+m)!}\right\}$$

$$= \frac{u_m(2\ell-1)(\ell+m-1)}{(\ell-m)^2}\frac{(\ell-m)!}{(\ell+m)!}$$

and

$$g_3 = \frac{2\ell-1}{\ell}\left\{\frac{u_m-1}{(\ell-m)^2}\frac{(\ell-m)!}{(\ell+m-2)!} + \tfrac{1}{2}u_{m+1}(\ell+m-1)^2\frac{(\ell-m-2)!}{(\ell+m)!}\right\} - \frac{\ell-1}{\ell}u_m\frac{(\ell-m-2)!}{(\ell+m-2)!}$$

$$= \frac{u_m(\ell+m-1)^2}{(\ell-m)^2}\frac{(\ell-m)!}{(\ell+m)!} .$$

Hence

$$f(\theta', \theta, \ell, m) = u_m\frac{(\ell-m)!}{(\ell+m)!}\frac{(2\ell-1)\cos\theta' P^m_{\ell-1}(\cos\theta') - (\ell+m-1)P^m_{\ell-2}(\cos\theta')}{\ell-m}$$

$$\times \frac{(2\ell-1)\cos\theta P^m_{\ell-1}(\cos\theta) - (\ell+m-1)P^m_{\ell-2}(\cos\theta)}{\ell-m}$$

$$= u_m\frac{(\ell-m)!}{(\ell+m)!}P^m_\ell(\cos\theta')P^m_\ell(\cos\theta)$$

on using equation (D5).

This completes the proof of the addition theorem.

Appendix D

RECURRENCE RELATIONS

The purpose of this Appendix is to prove the important relations, for the A, B, C, D and E functions, given by equations (50) to (57) of the main text. The starting point is the definition of Legendre polynomials by means of the identity

$$(1-2\,xy+y^2)^{-\frac{1}{2}} \equiv \sum_{0}^{\infty} y^n\,P_n(x) \quad ,$$

from which it follows at once that

$$(1-2\,xy+y^2)\left(\sum_{0}^{\infty} y^n\,P_n(x)\right)^2 = 1 \quad .$$

On differentiating both sides partially with respect to x and y in turn the resulting pair of relations leads, after some manipulation, to the classical recurrence formulae

$$x\,P'_{n+1}(x) - P'_n(x) = (n+1)\,P_{n+1}(x) \tag{D1}$$

and

$$P'_{n+1}(x) - x\,P'_n(x) = (n+1)\,P_n(x) \tag{D2}$$

for $n > 0$.

These last two formulae may be differentiated m times with respect to x, leading to:

$$x\,P^{(m+1)}_{n+1}(x) - P^{(m+1)}_n(x) = (n-m+1)\,P^{(m)}_{n+1}(x) \tag{D3}$$

and

$$P^{(m+1)}_{n+1}(x) - x\,P^{(m+1)}_n(x) = (n+m+1)\,P^{(m)}_n(x)\,. \tag{D4}$$

We replace, in equation (D3), n by n-1 and, in both equations, m by m-1. After the elimination of $P^{(m-1)}_n(x)$ and the introduction of the Legendre-function notation by equation (B2), it then follows that

$$(n-m+1)\, P_{n+1}^{m}(x) \;-\; (2n+1)x\, P_{n}^{m}(x) \;+\; (n+m)\, P_{n-1}^{m}(x) \;=\; 0 \quad . \qquad (D5)$$

Equation (D5) is the key recurrence relation and holds for $0 < m < n-1$. However, a second relation, proved similarly, is also given, since it is used in Appendix C:

$$(1-x^2)^{\frac{1}{2}}\, P_{n}^{m+1}(x) \;=\; (n+m)\, P_{n-1}^{m}(x) \;-\; (n-m)x\, P_{n}^{m}(x) \quad . \qquad (D6)$$

We can now proceed to derive the required recurrence relations for the A, B, C, D and E functions.

From equations (17) and (18) of section 4.2 of the main text,

$$\sin^k i\, A_{\ell}^{k}(i) \;=\; 2^k \frac{(\ell-k)!}{(\ell+k)!}\, k!\; P_{\ell}^{k}(\cos i) \quad . \qquad (D7)$$

Hence from equation (D5), with $n = \ell-1$ and $m = k$,

$$(\ell-k)\frac{(\ell+k)!}{(\ell-k)!}A_{\ell}^{k}(i) \;-\; (2\ell-1)\frac{(\ell+k-1)!}{(\ell-k-1)!}\cos i\, A_{\ell-1}^{k}(i)$$

$$+\; (\ell+k-1)\frac{(\ell+k-2)!}{(\ell-k-2)!}A_{\ell-2}^{k}(i) \;=\; 0 \quad .$$

The required relation for $A_{\ell}^{k}(i)$ follows at once and is given by equation (50) of section 6.2.

For the $B_h^k(e)$ function, it follows from equation (31) of section 4.2 and equation (E13) of Appendix E that

$$B_h^k(e) \;=\; \frac{k!\;\;(h-k-1)!}{(h+k-1)!}\left(\frac{e}{2}\right)^{-k}\zeta^{1-h}\,(-1)^{3k/2}\,P_{h-1}^{k}(\zeta) \quad , \qquad (D8)$$

where $\zeta = (1-e^2)^{-\frac{1}{2}}$. Hence from equation (D5), with $n = h-2$ and $m = k$,

$$(h-k-1)\frac{(h+k-1)!}{(h-k-1)!}\zeta^{h-1}B_h^k(e) \;-\; (2h-3)\frac{(h+k-2)!}{(h-k-2)!}\zeta^{h-1}B_{h-1}^{k}(e)$$

$$+\; (h+k-2)\frac{(h+k-3)!}{(h-k-3)!}\zeta^{h-3}B_{h-2}^{k}(e) \;=\; 0 \quad .$$

The required relation for $B_h^k(e)$ follows at once and is given by equation (51) of section 6.2.

For C_n^k (with $\ell = n$ if $n > 0$ and $\ell = -n-1$ if $n < 0$) we have, from equation (34) of section 4.2 and equation (B3) of Appendix B, that if $\ell-k$ is even,

$$C_n^k = (-1)^{\frac{1}{2}(\ell-k)+1} \frac{u_k \, (\ell+k)! \, (n-1)(n-2) \, \ldots. \, (n-k)}{2^{2k+\ell} \, (k!)^2 \, \{\frac{1}{2}(\ell-k)\}! \, \{\frac{1}{2}(\ell+k)\}!} , \qquad \text{(D9)}$$

while if $\ell-k$ is odd,

$$C_n^k = 0 . \qquad \text{(D10)}$$

Considering only the case $\ell-k$ even, we have, by dealing with the 'n factors' and the 'ℓ factors' separately, from the 'n factors' a contribution to C_n^k/C_{n-2}^k of

$$\frac{-(n-1)(n-2)}{(n-k-1)(n-k-2)}$$

and from the 'ℓ factors' a contribution to $C_\ell^k/C_{\ell-2}^k$ or $C_{-\ell-1}^k/C_{-\ell+1}^k$ (according as $n = \ell$ or $n = -\ell-1$) of

$$\frac{\ell+k-1}{\ell-k} .$$

Hence if $n = \ell$ (i.e. for $n > 0$),

$$C_n^k/C_{n-2}^k = -\frac{(n-1)(n-2)(n+k-1)}{(n-k)(n-k-1)(n-k-2)}$$

and if $n = -\ell-1$ (i.e. for $n < 0$),

$$C_n^k/C_{n+2}^k = -\frac{(-n+k-2)(-n+k-1)(-n+k)}{(-n)(-n-1)(-n-k-1)} .$$

These are the same as equations (52) and (54).

From equation (D9) it follows that, for $k > 0$,

$$C_{k+2}^k = \frac{u_k \, (2k+2)!}{2^{3k+2}(k!)^2} \qquad \text{and} \qquad C_{-k-1}^k = (-1)^{k-1} \frac{u_k (2k)! \, (2k+1)!}{2^{3k}(k!)^3 \, (k+1)!} ,$$

from which equations (53) and (55) may be derived at once.

For the function $D_\ell^k(i)$, there is no three-term recurrence relation when $k \geq 1$. Such a relation exists for the case $k = 0$, however, and this case is the most important. From equation (46),

$$D_\ell^0(i) = -f \frac{d\, P_\ell(\cos i)}{d(\cos i)}$$

$$= -\sin i\, P_\ell^1(\cos i) \quad . \qquad \text{(D11)}$$

The required relation follows at once, using equation (D5), and is given by equation (56). It is useful to note that the leading term of $D_\ell^0(i)$ is $-\frac{1}{2}\ell(\ell+1)f \cos i$ if ℓ is even, and $-\frac{1}{2}\ell(\ell+1)f$ if ℓ is odd; the leading term of $D_\ell^k(i)$, for $k \geq 1$, is $k \cos i$ if $\ell-k$ is even, and k if $\ell-k$ is odd.

For $E_n^k(e)$, there is again no three-term recurrence relation unless $k = 0$. But it is shown in Appendix E that for $n \geq 0$

$$E_n^0(e) = -(-1)^{\frac{1}{2}}\, e(1-e^2)^{\frac{1}{2}n}\, P_n^1(\zeta) \quad . \qquad \text{(D12)}$$

Hence $E_n^0(e)$ satisfies a relation similar to that which holds for $D_\ell^0(i)$; it is given by equation (57).

For $n < 0$ there is no need to use a recurrence relation; for it follows from Appendix E that, when $n < 0$, $E_n^0(e)$ is identical with $E_{-n-1}^0(e)$. It must be noted that there is no similar result when $k \geq 1$.

Leading terms for $E_n^k(e)$ are $\frac{1}{2}n(n+1)e^2$ for $k = 0$, $n > 0$, and k for $k \geq 1$ (and any n).

Appendix E

PROPERTIES OF THE FUNCTIONS $S_{nk}(e)$ AND $B_n^k(e)$

The main result of this Appendix is the establishment of equation (32) of the main text, relating $S_{nk}(e)$, for $n < 0$, with $B^k_{-n+1}(e)$. After this, simple mathematical expressions for $S_{nk}(e)$ are obtained, covering all values of n. Finally, two relations for $E^o_n(e)$ are obtained, those given by equation (D12) of Appendix D, and equation (58) of the main text.

Instead of proving equation (32) itself, it is convenient, first, to extend the definition of $B^k_n(e)$ to cover $n < 0$. In this case $\binom{n-1}{k}$, equal to $(n-1)(n-2) \ldots (n-k)/k!$, is non-zero and we define $B^k_n(e)$ by the first of equations (31). For the purpose of symmetry in expressing the result, we use a slightly different range of k, employing h(=n if $n > 0$ and $-n+1$ if $n < 0$). Then the result to be proved, for all n and $k < h-1$, is

$$B^k_n(e) \quad = \quad (1-e^2)^{n-\frac{1}{2}} \, B^k_{-n+1}(e) \quad . \tag{E1}$$

Since the starting point is the function $S_{nk}(e)$, which is defined only as a series, the proof is not short. We divide it into stages and prove four lemmas.

Lemma 1
$$B^k_n(e) \quad = \quad F\left(\frac{-n+k+1}{2} \; , \quad \frac{-n+k+2}{2} \; ; \quad k+1 \; ; \quad e^2\right) \tag{E2}$$

for $k < h-1$, where $F(a, b; c; x)$ is the hypergeometric series defined by

$$F(a,\ b;\ c;\ x) \quad = \quad 1 + \frac{a.b}{c.1}\, x + \frac{a(a+1)\ b(b+1)}{c(c+1)\ 1.2}\, x^2 + \ldots$$

$$= \quad \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \sum_{q=0}^{\infty} \frac{\Gamma(a+q)\ \Gamma(b+q)}{\Gamma(c+q)\ q!}\, x^q$$

and the gamma function $\Gamma(x)$ is related to the factorial function by $\Gamma(x) \; = \; (x-1)!$

This lemma can be proved by considering the general term ω_q of the series $S_{nk}(e)$ given by equation (29). We have

$$\omega_q = \frac{(n-1)(n-2)\;....\;(n-k-2q)}{(k+2q)!}\;\frac{(k+2q)!}{(k+q)!\;\;q!}\left(\frac{e}{2}\right)^{2q} \tag{E3}$$

$$= \frac{(n-1)\;....\;(n-k)}{k!}\;\frac{-n+k+1}{2}\;\frac{-n+k+2}{2}\;....\;\frac{-n+k+2q}{2}\;\frac{k!\;e^{2q}}{(k+q)!\;q!}$$

Taking the factors $\frac{1}{2}(-n+k+1)$, $\frac{1}{2}(-n+k+2)$ etc alternately in two sets we may now write

$$\omega_q = \binom{n-1}{k}\frac{\Gamma\left(\frac{-n+k+1}{2}+q\right)\Gamma\left(\frac{-n+k+2}{2}+q\right)}{\Gamma\left(\frac{-n+k+1}{2}\right)\Gamma\left(\frac{-n+k+2}{2}\right)}\;\frac{\Gamma(k+1)}{\Gamma(k+1+q)}\;(e^2)^q\;;$$

i.e.

$$S_{nk}(e) = \binom{n-1}{k} F\left(\frac{-n+k+1}{2}\,,\;\frac{-n+k+2}{2}\;;\;k+1\;;\;e^2\right)\;, \tag{E4}$$

from which the result follows using equation (31).

<u>Lemma 2</u>

For an arbitrary set of m quantities v_1, v_2, ... v_m, where $0 \leqslant m \leqslant q-1$, we have

$$\sum_{s=0}^{q}(-1)^s\binom{q}{s}(v_1-s)(v_2-s)\;....\;(v_m-s) = 0\;. \tag{E5}$$

This lemma is easily proved by observing that, if $\omega_s = (v_1-s)(v_2-s)\;....\;(v_m-s)$, then ω_s is an mth degree polynomial in s. Since $q > m$ the qth order differences of this polynomial vanish. But these differences are simply $\sum_{s=0}^{q}(-1)^s\binom{q}{s}\omega_s$.

<u>Lemma 3</u>

If G(a, b, c, q) is defined by

$$G(a,\,b,\,c,\,q) = \sum_{s=0}^{q}\binom{q}{s} a(a+1)\;....\;(a+q-1-s)\;b(b+1)\;....\;(b+q-1-s)(c+q-s)\;....$$
$$(c+q-1)(c-a-b)\;....\;(c-a-b-1+s)\;, \tag{E6}$$

then

$$G(a,\,b,\,c,\,q) = (c-a)(c-a+1)\;....\;(c-a+q-1)(c-b)(c-b+1)\;....\;(c-b+q-1)\,. \tag{E7}$$

This lemma may be proved by checking that each factor on the R.H.S. of equation (E7) is a factor of G(a, b, c, q). The result will follow at once since there are the same number (2q) of factors on the R.H.S. as in each term of G(a, b, e, q) while the term c^{2q}, with unit coefficient, occurs on the R.H.S. of equation (E7) and in the last term of G(a, b, c, q).

To check that c-a+p, for $0 \leqslant p \leqslant q-1$, is a factor of G(a, b, c, q) we show that G(a, b, a-p,q) = 0. Now the q+1 factors a, a+1, ... a+q-1-p, b, b+1, ..., b+p occur in every term of G(a, b, a-p, q). Dividing them out, there remains the series

$$\sum_{s=0}^{q} (-1)^s \binom{q}{s} (a+q-p-s) \;....\; (a+q-1-s)(b+p+1-s) \;....\; (b+q-1-s)$$

which vanishes by lemma 2, with m = q-1. Thus c-a+p, and similarly c-b+p, is a factor of G(a, b, c, q).

<u>Lemma 4</u>

$$F(a,\ b;\ c;\ x) \times F(c-a-b,\ 1;\ 1;\ x) = F(c-b,\ c-a;\ c;\ x) \quad . \tag{E8}$$

On multiplying out the two series on the L.H.S., it may be seen that the coefficient of x^q is

$$\sum_{s=0}^{q} \frac{a(a+1) \;....\; (a+q-1-s)\ b(b+1) \;....\; (b+q-1-s)}{c(c+1) \;....\; (c+q-1-s)\ (q-s)!} \times \frac{(c-a-b)(c-a-b+1) \;....\; (c-a-b-1+s)}{s!} ,$$

which is simply

$$\frac{G(a,\ b,\ c,\ q)}{c(c+1) \;....\; (c+q-1)\ q!}$$

or again, from lemma 3,

$$\frac{(c-a)(c-a+1) \;....\; (c-a+q-1)(c-b)(c-b+1) \;....\; (c-b+q-1)}{c(c+1) \;....\; (c+q-1)\ q!} .$$

But this last quantity is the coefficient of x^q in F(c-b, c-a; c; x). This completes the proof of lemma 4.

The proof of the main result of this Appendix is now quite straightforward. In lemma 4 we take

$$a = \frac{-n+k+1}{2}, \quad b = \frac{-n+k+2}{2}, \quad c = k+1, \quad x = e^2 .$$

This gives

$$F\left(\frac{-n+k+1}{2}, \frac{-n+k+2}{2}; k+1; e^2\right) \times F(n-\tfrac{1}{2}, 1; 1; e^2) = F\left(\frac{n+k}{2}, \frac{n+k+1}{2}; k+1; e^2\right) . \qquad \text{.... (E9)}$$

But $F(n-\frac{1}{2}, 1; 1; e^2) = (1-e^2)^{-n+\frac{1}{2}}$; lemma 1 and equation (E9) then give

$$B_n^k(e) \times (1-e^2)^{-n+\frac{1}{2}} = B_{-n+1}^k(e) . \qquad \text{(E10)}$$

This is the result required.

Next, in this Appendix, we give explicit expressions for $S_{nk}(e)$ for all n and $k < \ell$.

First, if $n < 0$, then from equation (E3),

$$\omega_q = (-1)^k \frac{(-n+k+2q)!}{(-n)!} \frac{1}{(k+q)!\, q!} \left(\frac{e}{2}\right)^{2q}$$

$$= (-1)^k \frac{(2k+1)!}{k!(-n)!} \frac{(-n+k+2q)!}{(2k+1+2q)!} \frac{(2k+1+2q)!\, k!}{2^{2q}(2k+1)!\, q!(k+q)!} e^{2q} .$$

But

$$\frac{(2k+1+2q)!\, k!}{2^{2q}\,(2k+1)!\, q!(k+q)!} = \frac{(2k+3)(2k+5) \dots (2k+1+2q)}{2^q\, q!}$$

$$= \binom{k+\frac{1}{2}+q}{q}$$

and so

$$\omega_q = (-1)^k \frac{(2k+1)!}{k!(-n)!} \binom{k+\frac{1}{2}+q}{q} e^{-2k-1} \frac{d^{-n-k-1}\, e^{-n+k+2q}}{de^{-n-k-1}} ;$$

i.e.

$$S_{nk}(e) = (-1)^k \frac{(2k+1)!}{k!(-n)!} e^{-2k-1} \frac{d^{-n-k-1}\{e^{-n+k}(1-e^2)^{-k-1\frac{1}{2}}\}}{de^{-n-k-1}} . \qquad (E11)$$

If $n > 0$, $-n-k-1$ will be negative and the above formula breaks down. For $n = 0$, k is restricted to 0 and we have, easily,

$$S_{oo}(e) = (1-e^2)^{-\frac{1}{2}} . \qquad (E12)$$

Finally, if $n > 1$ we can prove a result obtained by Groves[3], in a different notation, viz.

$$S_{nk}(e) = \frac{(n-1)!}{(n+k-1)!}\left(\frac{e}{2}\right)^{-k} \zeta^{1-n} (-1)^{3k/2} P^k_{n-1}(\zeta) , \qquad (E13)$$

where $\zeta = (1-e^2)^{-\frac{1}{2}}$ (the factor $(-1)^{3k/2}$ conforms with a remark in Appendix B). We need a lemma:-

Lemma 5

$$F\left(\frac{-n+1+k}{2} + s, \frac{-n+2+k}{2} + s; k+1+s; 1\right) = \frac{(k+s)!(2n-2-2s)!}{2^{n-1-k-2s}(n+k-1)!(n-1-s)!} . \qquad (E14)$$

This follows fairly easily from Gauss's theorem for the hypergeometric function, viz[38]

$$F(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} , \qquad (E15)$$

on taking $a = \frac{-n+1+k}{2} + s$, $b = \frac{-n+2+k}{2} + s$, $c = k+1+s$. For then

$$F(\ldots\ \text{as in (E14)}\ \ldots) = \frac{(k+s)!\,(n-1\frac{1}{2}-s)(n-2\frac{1}{2}-s)\ \ldots\ \frac{1}{2}}{\frac{1}{2}(n+k-1)\ \frac{1}{2}(n+k-2)\ \frac{1}{2}(n+k-3)\ \ldots\ \frac{1}{2}}$$

$$= (k+s)!\,\frac{(2n-3-2s)!}{2^{n-3-2s}(n-2-s)!}\,\frac{2^{n+k-1}}{(n+k-1)!}$$

$$= \frac{(k+s)!\,(2n-2-2s)!}{2^{n-1-k-2s}(n+k-1)!(n-1-s)!} .$$

Having established the lemma we proceed to obtain the required expression for $S_{nk}(e)$. From equation (E3),

$$\omega_q = \frac{(n-1)!}{(n-1-k-2q)!(k+q)!\ q!\ 2^{2q}} \{1-(1-e^2)\}^q$$

and so

$$S_{nk}(e) = \sum_{q=0}^{\infty} \frac{(n-1)!}{(n-1-k-2q)!(k+q!)q!\ 2^{2q}} \sum_{s=0}^{q} \binom{q}{s} (-1)^s (1-e^2)^s \qquad \text{(E16)}$$

where we can take the q series as infinite, even though it terminates, since $(n-1-k-2q)! = \infty$ when $n-1-k-2q < 0$.

But

$$\sum_{q=0}^{\infty}\sum_{s=0}^{q} = \sum_{q,s}^{0\leqslant s\leqslant q} = \sum_{s=0}^{\infty}\sum_{q=s}^{\infty}$$

and hence

$$S_{nk}(e) = \sum_{s=0}^{\infty}\sum_{q=s}^{\infty} \frac{(-1)^s(n-1)!(1-e^2)^s}{(n-1-k-2q)!(k+q)!\ s!(q-s)!\ 2^{2q}}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s(n-1)!}{(n-1-k-2s)!(k+s)!\ s!\ 2^{2s}} \left\{\sum_{q=s}^{\infty} \frac{(n-1-k-2s)!(k+s)!}{2^{2q-2s}(n-1-k-2q)!(k+q)!(q-s)!}\right\} (1-e^2)^s.$$

.... (E17)

The quantity inside the central brackets may easily be seen to be $F\left(\frac{-n+1+k}{2} + s,\ \frac{-n+2+k}{2} + s;\ k+1+s;\ 1\right)$. So from lemma 5,

$$S_{nk}(e) = \sum_{s=0}^{\infty} \frac{(-1)^s (n-1)!(2n-2-2s)!(1-e^2)^s}{(n+k-1)!(n-1-k-2s)!(n-1-s)!\, s!2^{n-1-k}}$$

$$= \frac{(n-1)!2^k}{(n+k-1)!} \frac{\zeta^{k+1-n}}{2^{n-1}(n-1)!} \sum_{s=0}^{\infty} (-1)^s \frac{(2n-2-2s)!}{(n-1-k-2s)!} \binom{n-1}{s} \zeta^{n-1-k-2s}$$

$$= \frac{(n-1)!}{(n+k-1)!} \left(\frac{e}{2}\right)^{-k} \zeta^{1-n} \frac{(\zeta e)^k}{2^{n-1}(n-1)!} \frac{d^{n+k-1}\{(\zeta^2-1)^{n-1}\}}{d^{n+k-1}} \quad . \qquad \text{(E18)}$$

Since $\zeta^2 e^2 = \zeta^2-1$, the required expression for $S_{nk}(e)$ follows at once from equation (B3) of Appendix B, noting that $(\zeta^2-1)^{k/2} = (-1)^{3k/2}\,(1-\zeta^2)^{k/2}$.

The remaining results proved in this Appendix relate to the function $E_n^k(e)$ in the particular case when $k = 0$. First we obtain equation (D12) of Appendix D.

From equations (47), (31) and (E13), when $n > 0$,

$$E_n^o(e) = (2n-1)\, e^2\, B_n^o(e) + e(1-e^2)\, \frac{d}{de}\, B_n^o(e)$$

$$= e(1-e^2)^{n+\frac{1}{2}} \frac{d}{de} \{(1-e^2)^{\frac{1}{2}-n}\, B_n^o(e)\}$$

$$= e(1-e^2)^{n+\frac{1}{2}} \frac{d}{de} \{\zeta^n\, P_{n-1}(\zeta)\} \quad . \qquad \text{(E19)}$$

But $d\zeta/de = e\zeta^3$ since $\zeta = (1-e^2)^{-\frac{1}{2}}$; also, from equation (D2), with n replaced by $n-1$,

$$\zeta^n\, P'_{n-1}(\zeta) + n\,\zeta^{n-1}\, P_{n-1}(\zeta) = \zeta^{n-1}\, P'_n \quad .$$

Hence

$$E_n^o(e) = e^2(1-e^2)^{\frac{1}{2}(n-1)} P'_n(\zeta) \quad . \qquad \text{(E20)}$$

Using equation (B2) to introduce $P_n^1(\zeta)$, the result given by equation (D12) follows at once.

The other result - equation (58) of section 6 - is best proved by expressing $E_n^o(e)$, for $n < 0$, in terms of $P^1_{-n-1}(\zeta)$, i.e. in terms of $P^1_{h-2}(\zeta)$.

From equations (48) and (E13), if $n < 0$,

$$E_n^o(e) = e \frac{d}{de} B_h^o(e)$$

$$= e \frac{d}{de} \{\zeta^{1-h} P_{h-1}(\zeta)\} \quad . \qquad (E21)$$

But $d\zeta/de = e\zeta^3$ and from equation (D1), with $n = h-2$,

$$\zeta^{1-h} P'_{h-1}(\zeta) + (1-h)\, \zeta^{-h} P_{h-1}(\zeta) = \zeta^{-h} P'_{h-2}(\zeta) \quad . \qquad (E22)$$

Hence

$$E_n^o(e) = e^2(1-e^2)^{\frac{1}{2}(h-3)} P'_{h-2}(\zeta)$$

$$= -(-1)^{\frac{1}{2}}\, e(1-e^2)^{\frac{1}{2}(h-2)}\, P^1_{h-2}(\zeta)$$

$$= E^o_{h-2}(e) \qquad (E23)$$

by equation (D12), just proved. This is the result required.

Appendix F

AXIS TRANSFORMATION RELATIONS REQUIRED BY THE THEORY OF PERTURBATIONS DUE TO A STATIONARY DISTURBING BODY

In this Appendix we obtain the formulae required by section 8.3. Four systems of axes are relevant and are illustrated in Fig. 3. The fixed system Oxyz has been defined in section 8.3, Ox pointing to some instantaneous position of the node and Oz being perpendicular to the orbital plane. The system OXYZ relates to the (true) equator, OZ being towards the north pole and OX coincident with Ox. Since it is only the perturbation in Ω that is of interest, its origin is irrelevant and there is no need to introduce the direction of the first point of Aries. The systems Ox'y'z' and OX'Y'Z' are defined in an exactly analogous manner but relate to the pseudo-equatorial plane which is perpendicular to the direction of the fixed disturbing body.

Then

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \underset{\sim}{R}_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\sim}{R}_2 \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underset{\sim}{R}_3 \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$$

where

$$\underset{\sim}{R}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} , \quad \underset{\sim}{R}_2 = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\underset{\sim}{R}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i' & \sin i' \\ 0 & -\sin i' & \cos i' \end{pmatrix} .$$

We can at once derive equation (74) of section 8.3. For A, B, C are the direction cosines, in the Oxyz system, of the Z' axis, so that

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \underset{\sim}{R}_2 \, \underset{\sim}{R}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \qquad \text{(F1)}$$

To obtain equations (75), (76) and (77) we observe that the (small) changes Δi, $\Delta\Omega$ and $\Delta\omega$ represent the rotation of the orbital plane from one orientation to another. The total rotation is in fact compounded of rotations Δi about Ox, $\Delta\Omega$ about OZ and $\Delta\omega$ about Oz and hence has components, in the Oxyz system, given by (Δi, $\Delta\Omega \sin i$, $\Delta\Omega \cos i + \Delta\omega$), using $\underset{\sim}{R}_1$. In the Ox'y'z' system the components of the total rotation are, similarly, given by ($\Delta i'$, $\Delta\Omega' \sin i'$, $\Delta\Omega' \cos i' + \Delta\omega'$).

The preceding paragraph depends on the fact that small rotations are vectors. Using this fact again we get:-

$$\begin{pmatrix} \Delta i \\ \sin i \ \Delta\Omega \\ \Delta\omega + \Delta\Omega \cos i \end{pmatrix} = \underset{\sim}{R}_2 \begin{pmatrix} \Delta i' \\ \sin i' \ \Delta\Omega' \\ \Delta\omega' + \Delta\Omega' \cos i' \end{pmatrix} . \tag{F2}$$

Equations (75), (76) and (77) follow at once, an introducing A, B and C from equation (74).

Appendix G

RELATIONS REQUIRED BY THE THEORY OF PERTURBATIONS DUE TO A DISTURBING BODY IN A KEPLER ORBIT

The relations required are those given by equations (126) to (135) of section 8.4. It is convenient to define altogether six systems of axes and these are illustrated diagrammatically in Fig.4. The systems OXYZ and $OX_dY_dZ_d$ are based on the equatorial plane, intersected by the satellite orbit and the disturbing body orbit respectively. The systems Oxyz and Ox"y"z" are based on the satellite orbital plane, intersected by the equator and the disturbing body orbit respectively. The systems $Ox_dy_dz_d$ and OX"Y"Z" are based on the orbital plane of the disturbing body, intersected by the equator and the satellite orbit respectively.

All the relations derive from transformations between these systems of axes. As in Appendix F, the first three - given by equations (126) to (128) - follow from expressing the angular momentum vector for the satellite orbit in each of the systems Oxyz and Ox"y"z", and transforming. Thus

$$\begin{pmatrix} \dfrac{d\bar{i}}{dt} \\ \sin i \dfrac{d\bar{\Omega}}{dt} \\ \dfrac{d\bar{\omega}}{dt} + \cos i \dfrac{d\bar{\Omega}}{dt} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{d\bar{i}''}{dt} \\ \sin i'' \dfrac{d\bar{\Omega}''}{dt} \\ \dfrac{d\bar{\omega}''}{dt} + \cos i'' \dfrac{d\bar{\Omega}''}{dt} \end{pmatrix} .$$

The important relation given by equation (129) expresses the transformation of axes between systems Oxyz and $Ox_dy_dz_d$, carried out via the auxiliary systems OXYZ and $OX_dY_dZ_d$. The relation follows from the equations for these auxiliary transformations, viz.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} ,$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos(\Omega-\Omega_d) & \sin(\Omega-\Omega_d) & 0 \\ -\sin(\Omega-\Omega_d) & \cos(\Omega-\Omega_d) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix}$$

and

$$\begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i_d & -\sin i_d \\ 0 & \sin i_d & \cos i_d \end{pmatrix} \begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix} .$$

As in section 8.4 we abbreviate the full transformation, obtained by multiplying out the above matrixes, to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{pmatrix} \begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix} . \qquad \text{(G1)}$$

Equation (130) of section 8.4, viz $\omega'' = \omega - \alpha$, is immediate from the definition of α as the angle in the orbital plane of the satellite between Ox and Ox".

To obtain equations (131) to (133), we express the direction cosines of the axis Oz_d, identical with the axis OZ", in the system Oxyz by two different methods: first, using equation (G1), and second, transforming via the system Ox"y"z". Thus

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i'' & \sin i'' \\ 0 & -\sin i'' & \cos i'' \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin i'' \sin\alpha \\ \sin i'' \cos\alpha \\ \cos i'' \end{pmatrix} .$$

Finally, to obtain equations (134) and (135) we express z, identical with z", in terms of x_d, y_d, z_d in two different ways: first, using equation (G1), and second, transforming via the system OX"Y"Z". Then

$$(L_3 \quad M_3 \quad N_3) = (0 \quad -\sin i'' \quad \cos i'') \begin{pmatrix} \cos\Omega'' & \sin\Omega'' & 0 \\ -\sin\Omega'' & \cos\Omega'' & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

On taking components, we have (in addition to a second proof of equation (133))

$$L_3 = \sin i'' \sin \Omega''$$

and

$$M_3 = -\sin i'' \cos \Omega'' \quad .$$

But

$$\Omega'' = \omega_d - \theta_d$$

and hence

$$L_3 \cos \omega_d + M_3 \sin \omega_d = -\sin i'' \sin \theta_d$$

and

$$L_3 \sin \omega_d - M_3 \cos \omega_d = \sin i'' \cos \theta_d \quad .$$

These are the same as equations (134) and (135).

Table 1

The function $A_{\ell}^{k}(i)$

ℓ \ k	0	1	2	3	4
0	1				
1	$\cos i$	1			
2	$1-\frac{3}{2}f$	$\cos i$	1		
3	$\cos i\left(1-\frac{5}{2}f\right)$	$1-\frac{5}{4}f$	$\cos i$	1	
4	$1-5f+\frac{35}{8}f^2$	$\cos i\left(1-\frac{7}{4}f\right)$	$1-\frac{7}{6}f$	$\cos i$	1

Table 2

The function $B_{h}^{k}(e)$

h \ k	0	1	2	3	4	5
1	1	0				
2	1	1	0			
3	$1+\frac{1}{2}e^2$	1	1	0		
4	$1+\frac{3}{2}e^2$	$1+\frac{1}{4}e^2$	1	1	0	
5	$1+3e^2+\frac{3}{8}e^4$	$1+\frac{3}{4}e^2$	$1+\frac{1}{6}e^2$	1	1	0

Table 3

The coefficient C_n^k

n \ k	0	1	2	3	4
-4	0	$-\frac{15}{4}$	0	$\frac{175}{64}$	
-3	$\frac{1}{2}$	0	$-\frac{15}{8}$		
-2	0	$\frac{3}{2}$			
-1	-1				
0	-1				
1	0	0			
2	$\frac{1}{2}$	0	0		
3	0	$\frac{3}{2}$	0	0	
4	$-\frac{3}{8}$	0	$\frac{45}{32}$	0	0

Table 4

The function $D_\ell^k(i)$

ℓ \ k	0	1	2	3	4
0	0				
1	$-f$	$\cos i$			
2	$-3f\cos i$	$1-2f$	$2\cos i$		
3	$-6f+\frac{15}{2}f^2$	$\cos i\left(1-\frac{15}{4}f\right)$	$2-3f$	$3\cos i$	
4	$\cos i\left(-10f+\frac{35}{2}f^2\right)$	$1-\frac{29}{4}f+7f^2$	$\cos i\left(2-\frac{14}{3}f\right)$	$3-4f$	$4\cos i$

Table 5

The function $E_n^k(e)$

n \ k	0	1	2	3	4
-4	$6e^2 + \frac{3}{2} e^4$	$1 + \frac{9}{4} e^2$	$2 + \frac{2}{3} e^2$	3	
-3	$3e^2$	$1 + \frac{3}{4} e^2$	2		
-2	e^2	1			
-1	0				
0	0				
1	e^2	0			
2	$3e^2$	$1 + 2 e^2$	0		
3	$6e^2 + \frac{3}{2} e^4$	$1 + 4 e^2$	$2 + 3 e^2$	0	
4	$10e^2 + \frac{15}{2} e^4$	$1 + \frac{27}{4} e^2 + e^4$	$2 + 5 e^2$	$3 + 4 e^2$	0

SYMBOLS

Symbol	Section of first appearance	Definition
A,B,C	8.3	direction cosines of disturbing body in x,y,z axis system
$A_{\ell}^{k}(i)$	4.2	function of i given by equation (18)
$B_{h}^{k}(e)$	4.2	function of e given by equation (31) or (33)
C_{n}^{k}	4.2	constant given by equation (34)
$D_{\ell}^{k}(i)$	5	function of i given by equation (46)
$E_{n}^{k}(e)$	5	function of e given by equations (47) and (48)
F	9	solar radiation force per unit mass
$G_{\ell k}(e,i)$	10.2	given by $e_{\ell k}/\sin k\ (\omega-\frac{1}{2}\pi)$
$H_{\ell k}(e,i)$	10.2	given by $\Omega_{\ell k}/\cos k\ (\omega-\frac{1}{2}\pi)$
$I_{\ell k}(e,i)$	10.2	given by $\omega_{\ell k}/\cos k\ (\omega-\frac{1}{2}\pi)$
J_n	4.1	gravitational coefficients of asphericity
$K_{\ell k}(e,i)$	10.2	given by $\sigma_{\ell k}/\cos k\ (\omega-\frac{1}{2}\pi)$
$L_{\ell k}(e,i)$	10.2	given by $-3p_{\ell k}/2(\ell+1)\ \cos k\ (\omega-\frac{1}{2}\pi)$
L_1	8.4	$\cos\ (\Omega-\Omega_d)$
L_2	8.4	$\cos i_d \sin\ (\Omega-\Omega_d)$
L_3	8.4	$-\sin i_d \sin\ (\Omega-\Omega_d)$
M	3	mean anomaly
M_1	8.4	$-\cos i \sin\ (\Omega-\Omega_d)$
M_2	8.4	$\sin i_d \sin i + \cos i_d \cos i \cos\ (\Omega-\Omega_d)$
M_3	8.4	$\cos i_d \sin i - \sin i_d \cos i \cos\ (\Omega-\Omega_d)$
N_1	8.4	$\sin i \sin\ (\Omega-\Omega_d)$
N_2	8.4	$\sin i_d \cos i - \cos i_d \sin i \cos\ (\Omega-\Omega_d)$
N_3	8.4	$\cos i_d \cos i + \sin i_d \sin i \cos\ (\Omega-\Omega_d)$
$P_n(\ )$	4.1	Legendre polynomial
$P_n^m(\ )$	4.1	Legendre function
R	4.1	some fixed value of r, normally the mean equatorial radius of the earth or the distance to a disturbing body
$S_{nk}(e)$	4.2	function of e given by equation (29)
T	7	orbital period

SYMBOLS (Contd)

Symbol	Section of first appearance	Definition
U	3	disturbing function
U_n	4.1	term of U containing the factor r^{-n-1}
U_{nk}	4.1	term of U_n containing the factor $\cos k\gamma$
X,Y,Z	Appendix F	axes based on the equator, with OX towards the node of the satellite orbit
X',Y',Z'	Appendix F	axes based on the plane perpendicular to the direction of a fixed disturbing body as pseudo-equator, with OX' towards the satellite pseudo-node
X'',Y'',Z''	8.4	similar to X',Y',Z' but with the orbit of a disturbing body as pseudo-equator
a	3	semi-major axis
a'	7	constant related to a and U by equation (62)
b	4.2	a or p according as $n < 0$ or $n > 1$
d	8.2	suffix for disturbing body
e	3	eccentricity
f	4.2	$\sin^2 i$
h	4.2	$-n+1$ or n according as $n < 0$ or $n > 1$
i	3	orbital inclination
k	4.2	integer in the range $0 < k < \ell$
ℓ	4.1	$-n-1$ or n according as $n < 0$ or $n > 0$
n	3	mean motion
n	4.1	index associated with component U_n of U
n'	7	constant such that $n'^2 a'^3 = \mu$
p	4.2	$a(1-e^2)$
r	3	distance of satellite from centre of force
t	3	time
u_k	4.2	1 if $k = 0$, but 2 if $k > 0$
u_d	8.4	$\theta_d + v_d$
v	4.2	true anomaly
x,y,z	8.3	axes based on the satellite orbital plane, with Ox along OX
x',y',z'	Appendix F	axes based on the satellite orbital plane, with Ox' along OX'
x'',y'',z''	8.4	axes based on the satellite orbital plane, with Ox'' along OX''

SYMBOLS (Contd)

Symbol	Section of first appearance	Definition
α	8.3	angle between x and x' axes
β	4.1	geocentric latitude
γ	4.2	$\frac{1}{2}\pi - (\omega+v)$
Δ	5	perturbation for a complete satellite revolution
δ	10.1	perturbation other than for a complete satellite revolution
ε	4.2	0 if $n < 0$ but $\frac{1}{2}$ if $n > 1$
ζ	5	representative orbital element
ζ_n	5	given by equation (36)
ζ_{nk}	5	term of ζ_n associated with k
θ_d	8.4	$\omega_d - \Omega''$
μ	3	gravitational constant for primary centre of force
ρ	5	auxiliary element introduced by equation (43)
σ	3	modified mean anomaly at the epoch, used instead of χ
χ	3	mean anomaly at the epoch $t = 0$
ψ	5	auxiliary element introduced by equation (42)
Ω	3	right ascension of the ascending node
$\dot{\Omega}$	10.1	rate of secular motion of Ω associated with J_2
ω	3	argument of perigee
$\dot{\omega}$	10.1	rate of secular motion of ω associated with J_2
i', Ω', ω'	8.3	pseudo-elements associated with the axes X', Y', Z'
i'', Ω'', ω''	8.4	pseudo-elements associated with the axes X'', Y'', Z''

The above list does not include symbols introduced temporarily, with a limited use. Some of the symbols in the list have other uses of this type. The symbol n is used widely with both its meanings, but no confusion should arise.

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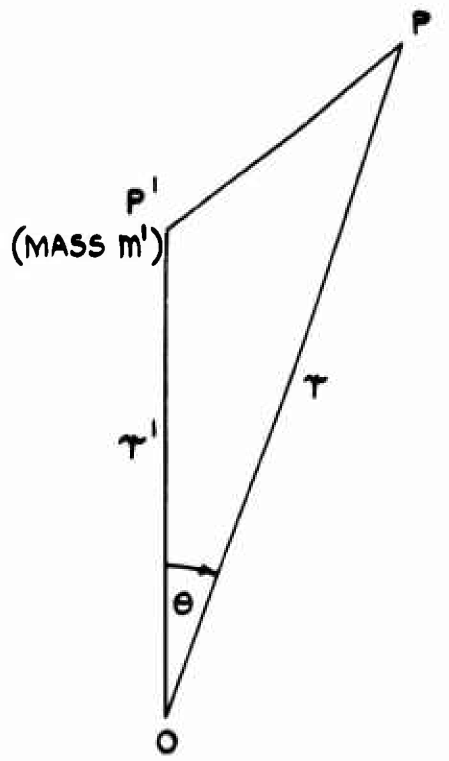


FIG. 1 (a) A SINGLE POINT MASS

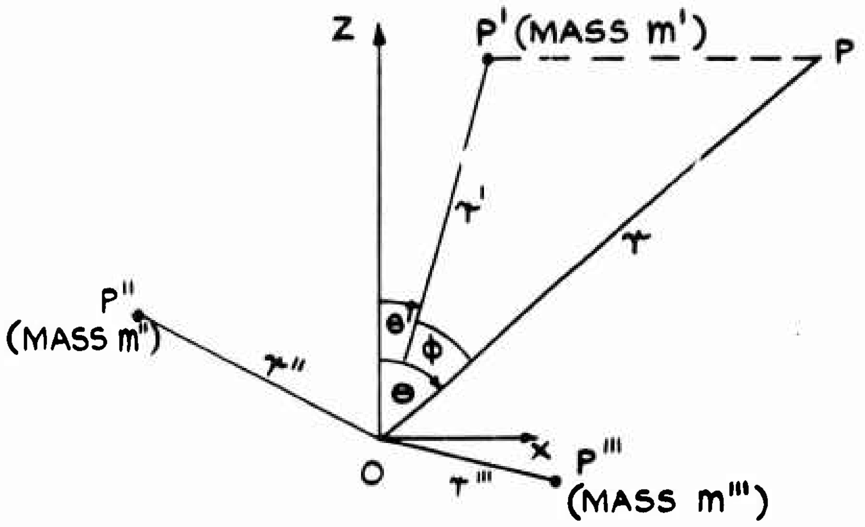


FIG. 1 (b) SEVERAL POINT MASSES

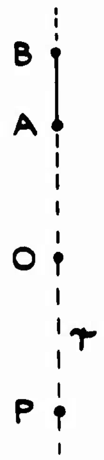


FIG. 1 (c) EXAMPLE WITH A LINE MASS

FIG. 1 SIMPLE MASS CONFIGURATIONS OF WHICH THE POTENTIAL IS STUDIED

Fig.2&3

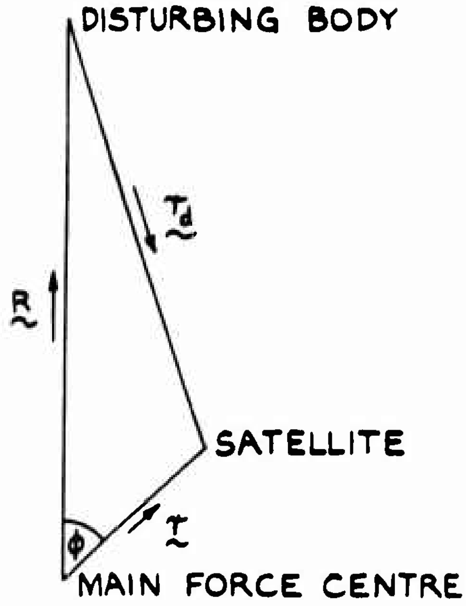
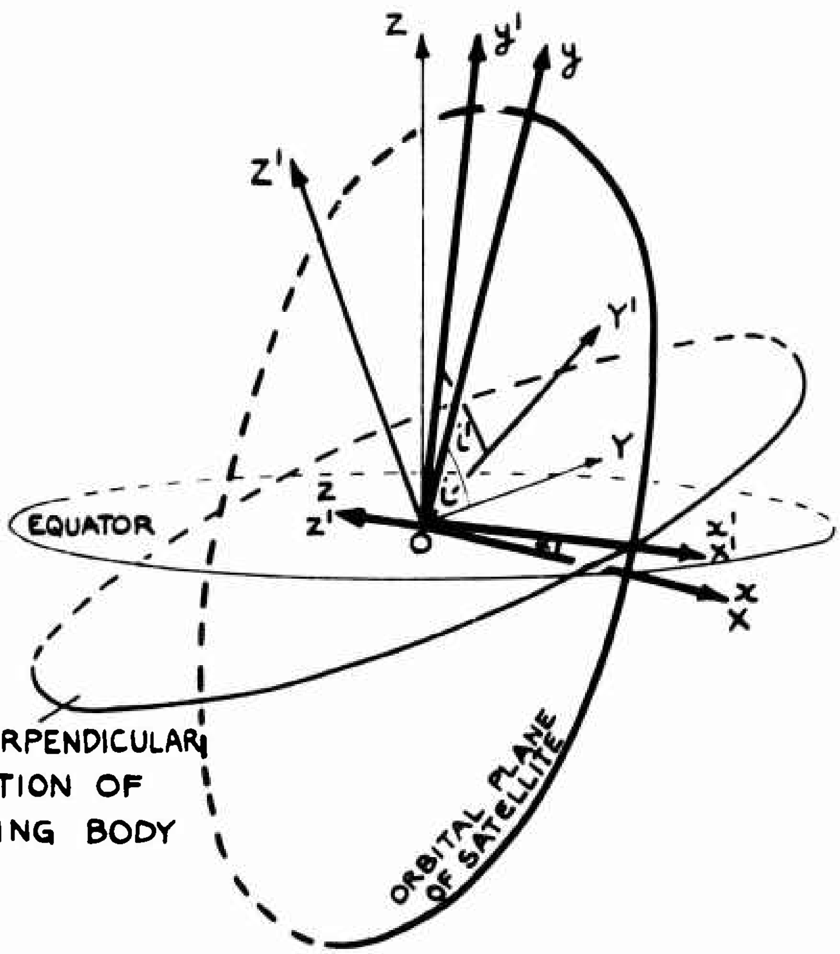


FIG.2 VECTORS FOR PERTURBATION EFFECTS DUE TO A DISTURBING BODY



OXYZ BASED ON EQUATOR WITH OX ALONG INTERSECTION WITH SATELLITE ORBIT PLANE

OX'Y'Z' BASED ON PLANE PERPENDICULAR TO DIRECTION OF DISTURBING BODY, OX' ALONG INTERSECTION WITH SATELLITE ORBIT PLANE

Oxyz BASED ON ORBITAL PLANE WITH Ox ALONG OX

Ox'y'z' BASED ON ORBITAL PLANE WITH Ox' ALONG OX'

FIG.3 AXIS SYSTEMS FOR THEORY OF PERTURBATIONS DUE TO A STATIONARY DISTURBING BODY

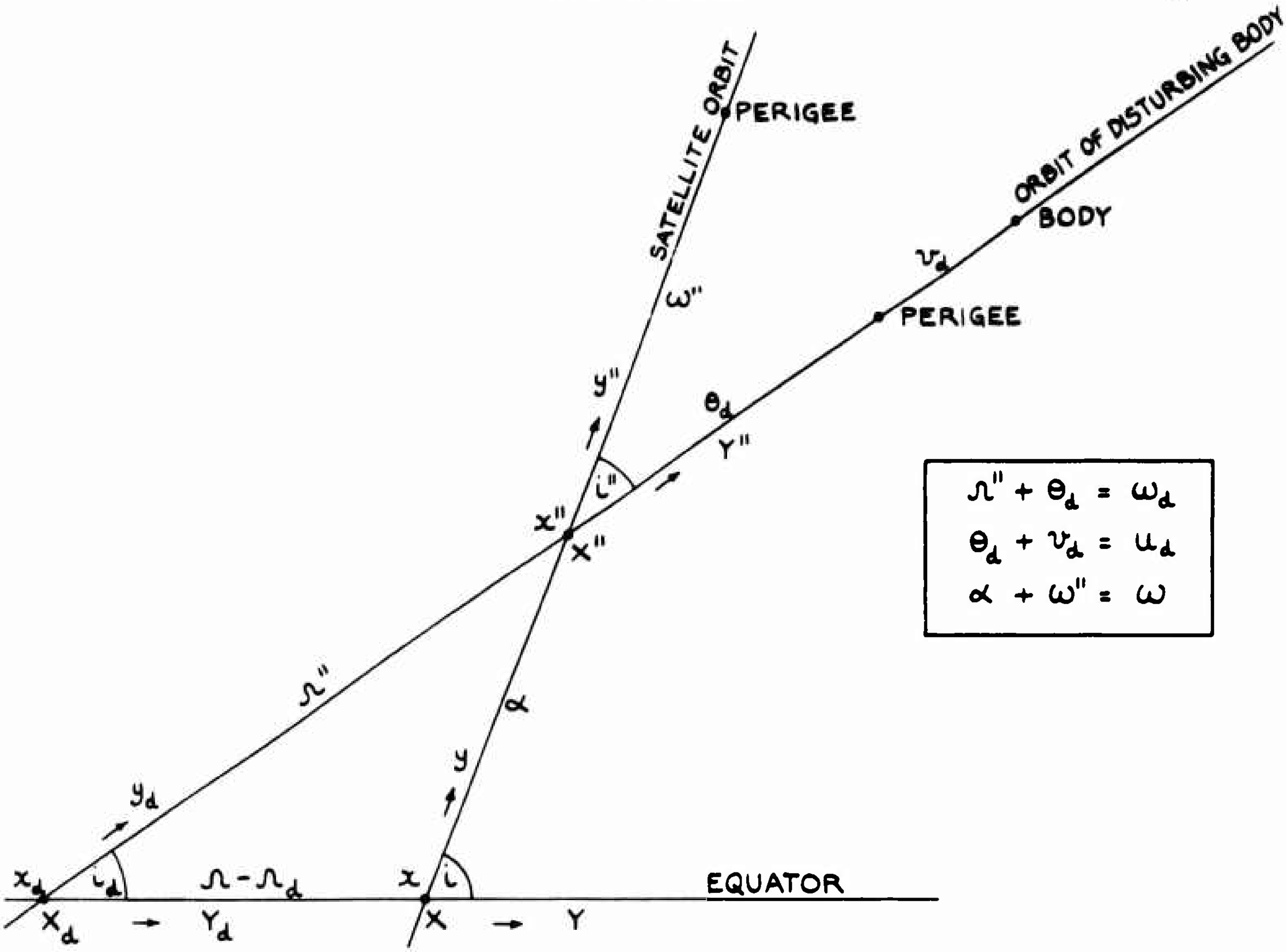


FIG.4 AXIS SYSTEMS ASSOCIATED WITH A DISTURBING BODY IN A KEPLER ORBIT

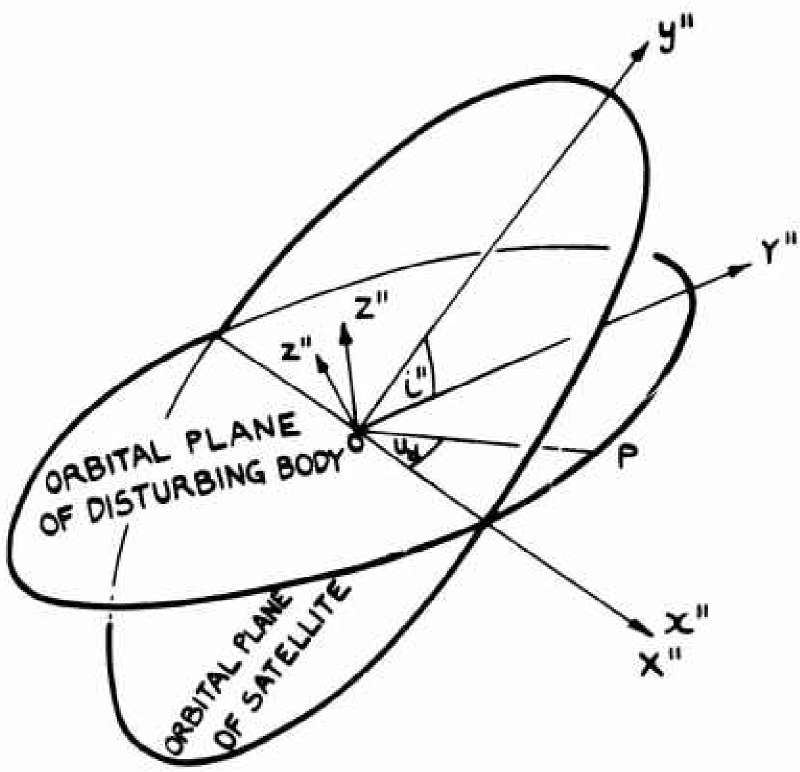


FIG.5 THE $Ox''y''z''$ AND $OX''Y''Z''$ AXIS SYSTEMS